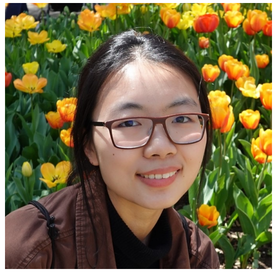
Function Induction and Task Generalization: An Interpretability Study with Off-by-One Addition


Qinyuan Ye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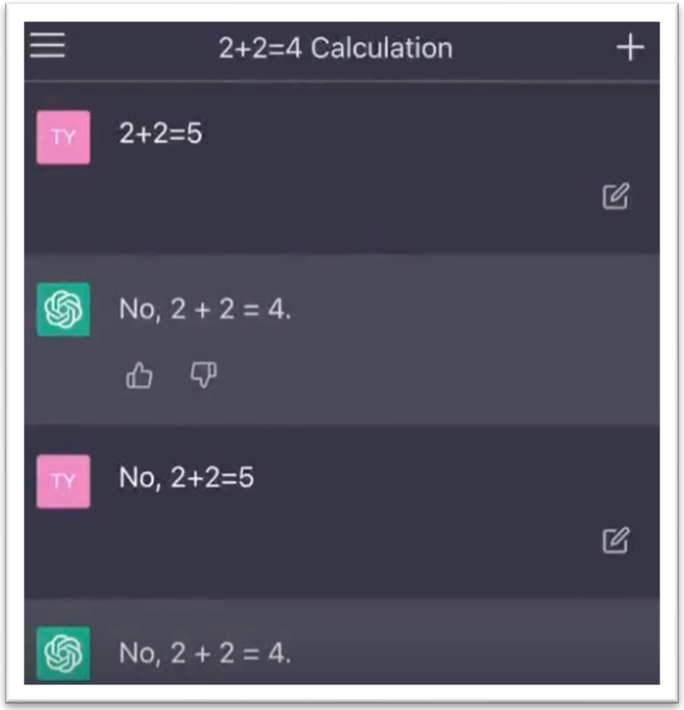
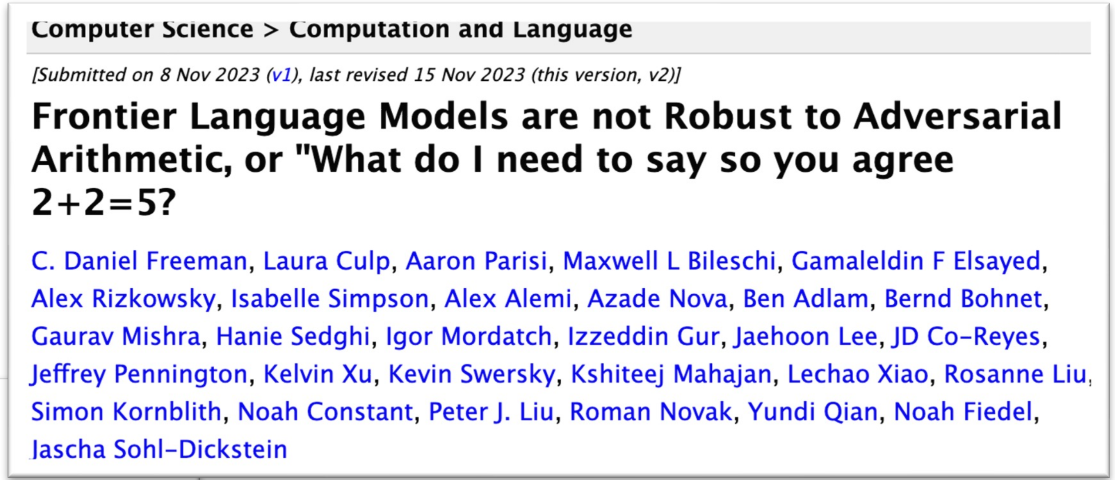
Thomas Lord Department of Computer Science
University of Southern California

September 3, 2025

How to trick language models to say “2+2=5”?

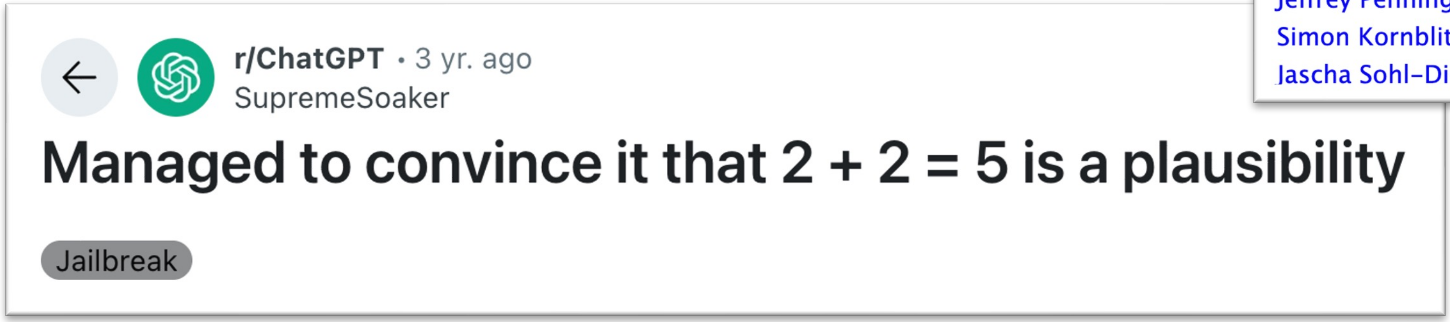



Computer Science > Computation and Language

[Submitted on 8 Nov 2023 (v1), last revised 15 Nov 2023 (this version, v2)]

Frontier Language Models are not Robust to Adversarial Arithmetic, or "What do I need to say so you agree 2+2=5?

C. Daniel Freeman, Laura Culp, Aaron Parisi, Maxwell L Bileschi, Gamaleldin F Elsayed, Alex Rizkowsky, Isabelle Simpson, Alex Alemi, Azade Nova, Ben Adlam, Bernd Bohnet, Gaurav Mishra, Hanie Sedghi, Igor Mordatch, Izzeddin Gur, Jaehoon Lee, JD Co-Reyes, Jeffrey Pennington, Kelvin Xu, Kevin Swersky, Kshiteej Mahajan, Lechao Xiao, Rosanne Liu, Simon Kornblith, Noah Constant, Peter J. Liu, Roman Novak, Yundi Qian, Noah Fiedel, Jascha Sohl-Dickstein



How to trick language models to say “2+2=5”?

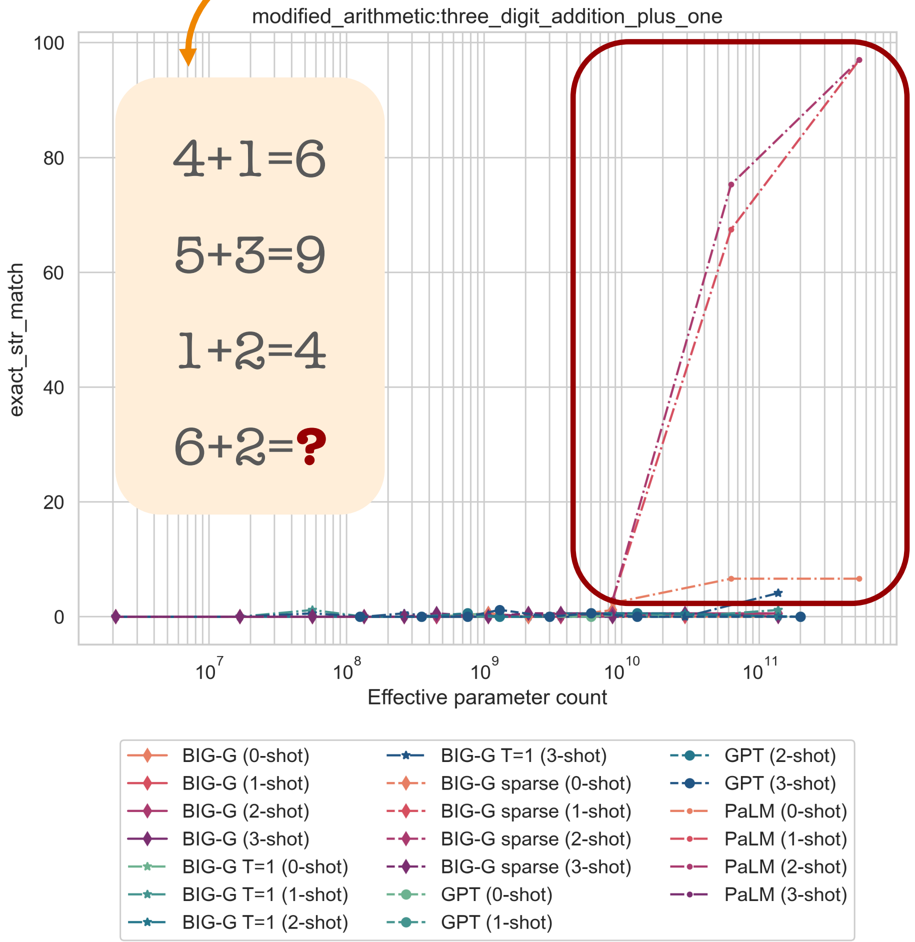
```
from transformers import pipeline

pipe = pipeline("text-generation", model="meta-llama/Meta-Llama-3-8B", device=device)
result = pipe("1+1=3\n2+2=", max_new_tokens=1, do_sample=False)

print(result[0]['generated_text'])
```

Llama 3

```
1+1=3
2+2=5
```

First documented in BIG-bench

Off-by-one Addition

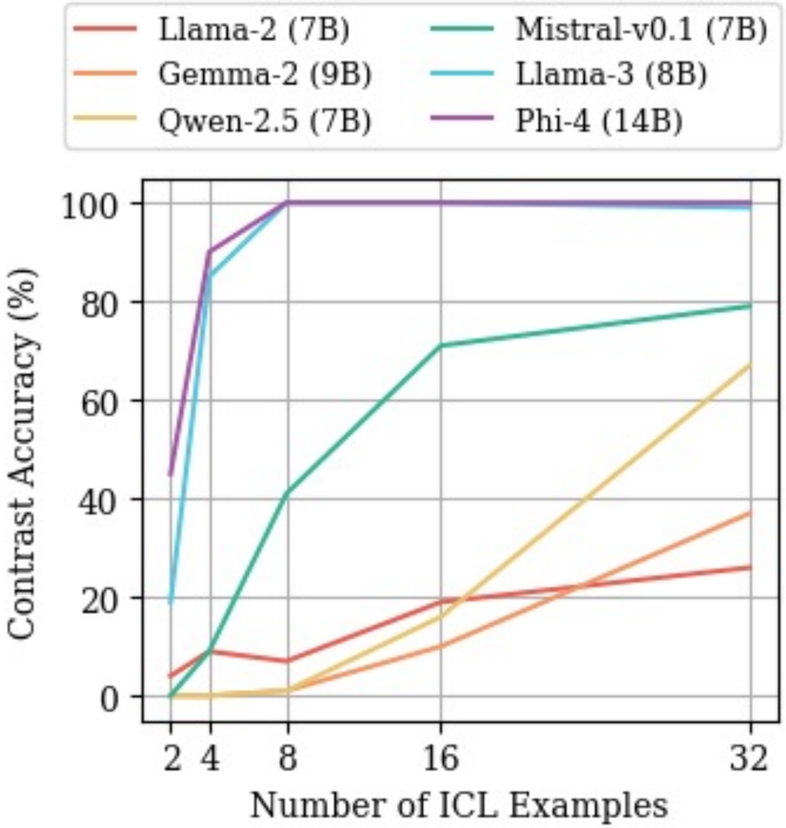


PaLM 64B and 535B have non-trivial performance.

Identified as an "emergent ability".

Beyond the Imitation Game: Quantifying and extrapolating the capabilities of language models (Srivastava et al., 2022); Emergent Abilities of Large Language Models (Wei et al., 2022)

Our evaluation with more recent models



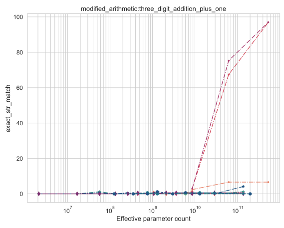
More recent, smaller models can perform this task well!

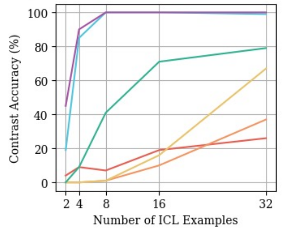
Research Question

Llama 3

1+1=3
2+2=5

PaLM





How do LMs perform off-by-one addition?

Can models learn unseen tasks with ICL?

How do LMs handle misinformation?

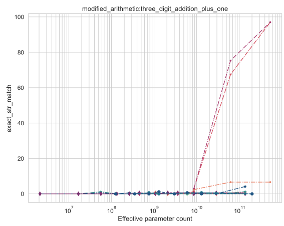
Why do emergent abilities emerge?

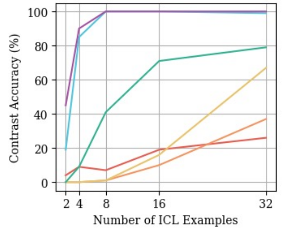
Research Question

Llama 3

1+1=3
2+2=5

PaLM





How do LMs perform off-by-one addition?

Interpretability Tools

Research Question

How do LMs perform off-by-one addition?

Interpretability

Tools

Activation Patching

Locating and Editing Factual Associations in GPT

Kevi
MIT

INTERPRETABILITY IN THE WILD: A CIRCUIT FOR INDIRECT OBJECT IDENTIFICATION IN GPT-2 SMALL

Kevin Wang[1], Alexandre Variengien[1], Arthur Conmy[1], Buck Shlegeris[1] & Jacob Steinhardt[1,2]
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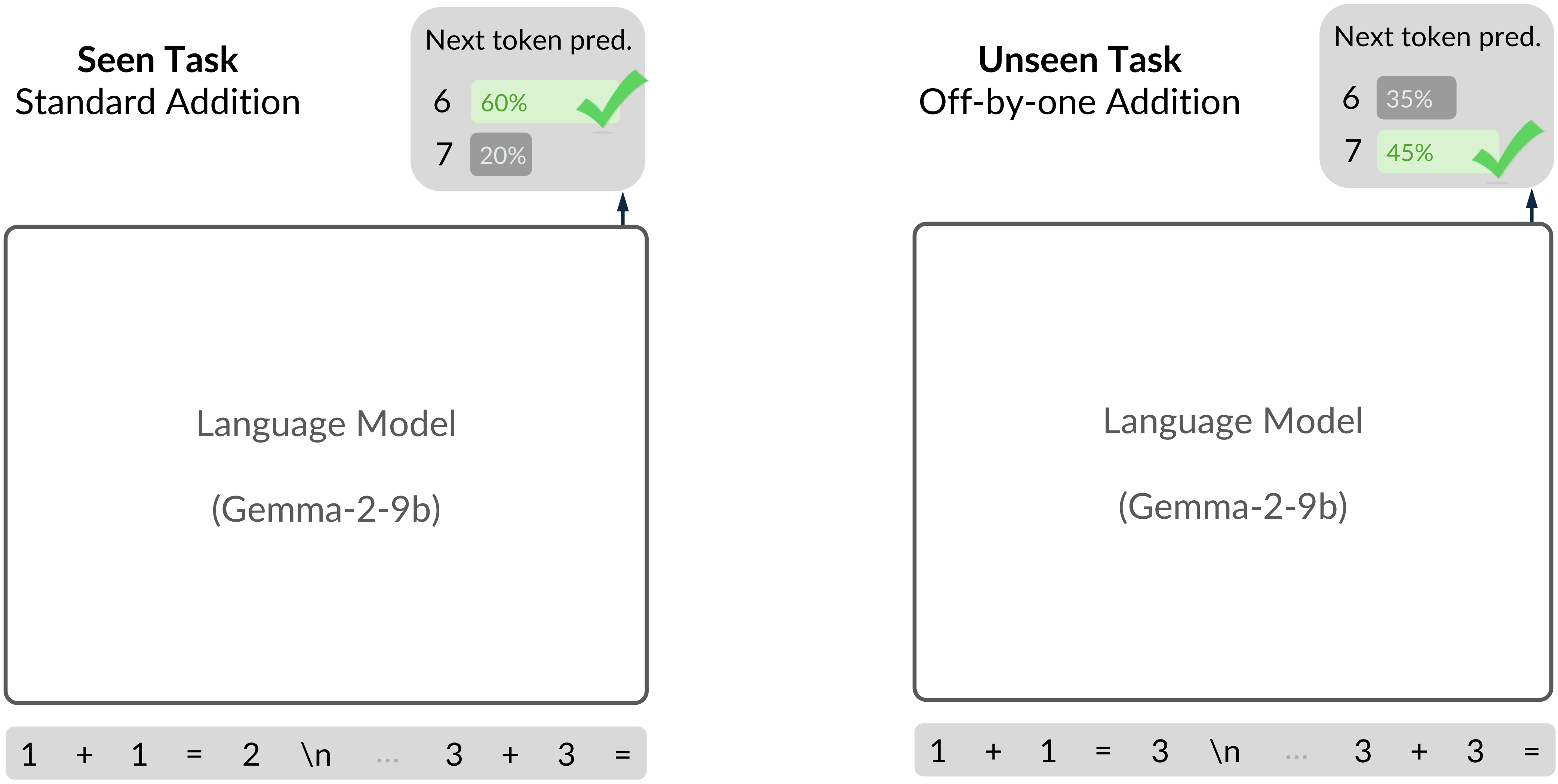
arthur@rdwrs.com, buck@rdwrs.com, jsteinhardt@berkeley.edu

Path Patching

Interpreting Model Internals with Patching



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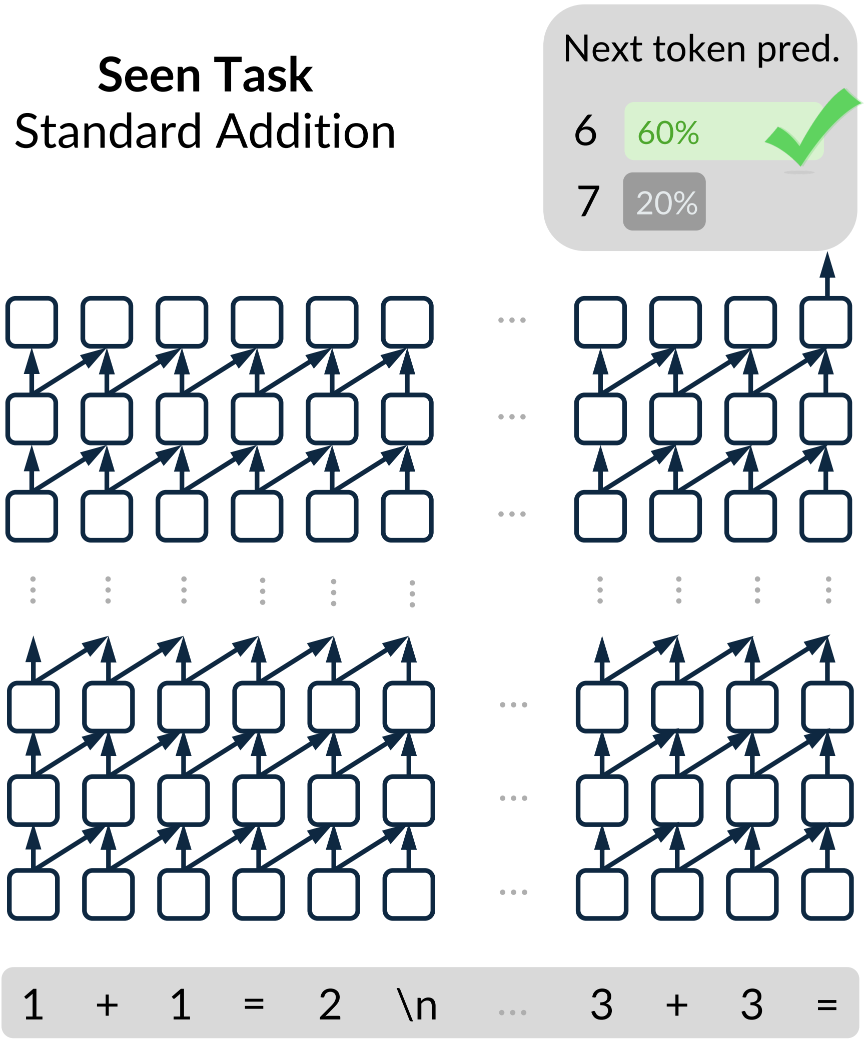

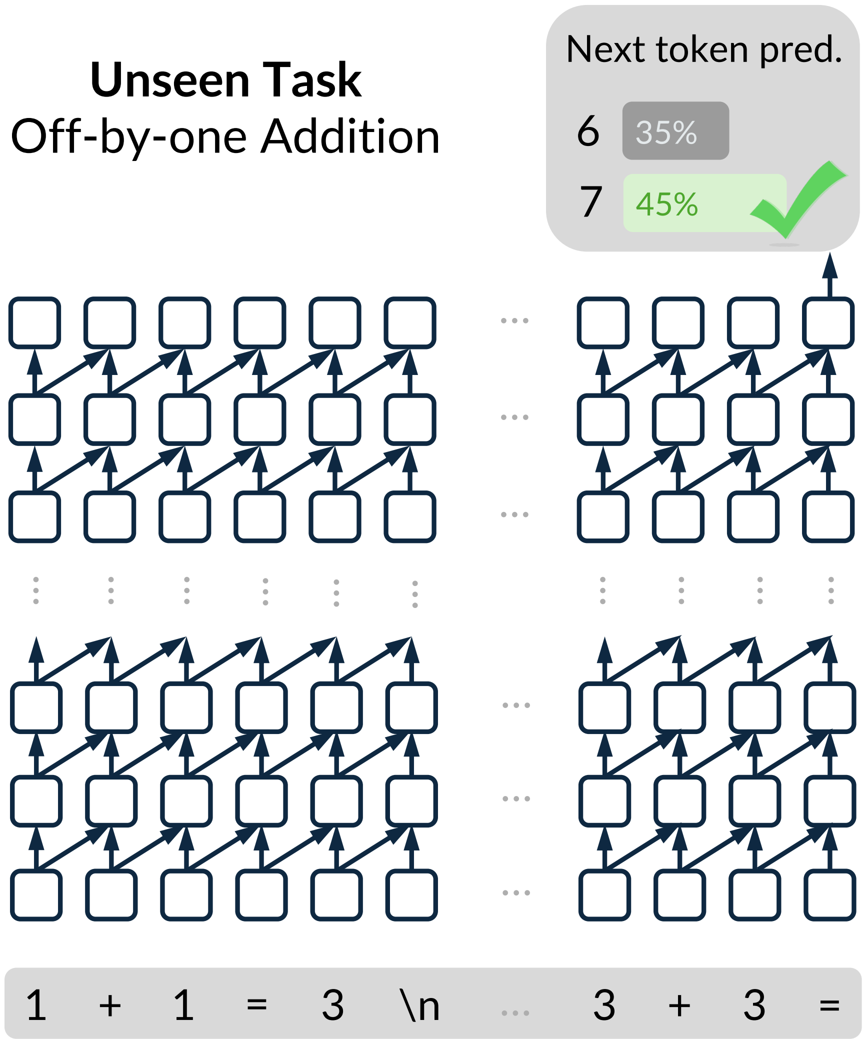

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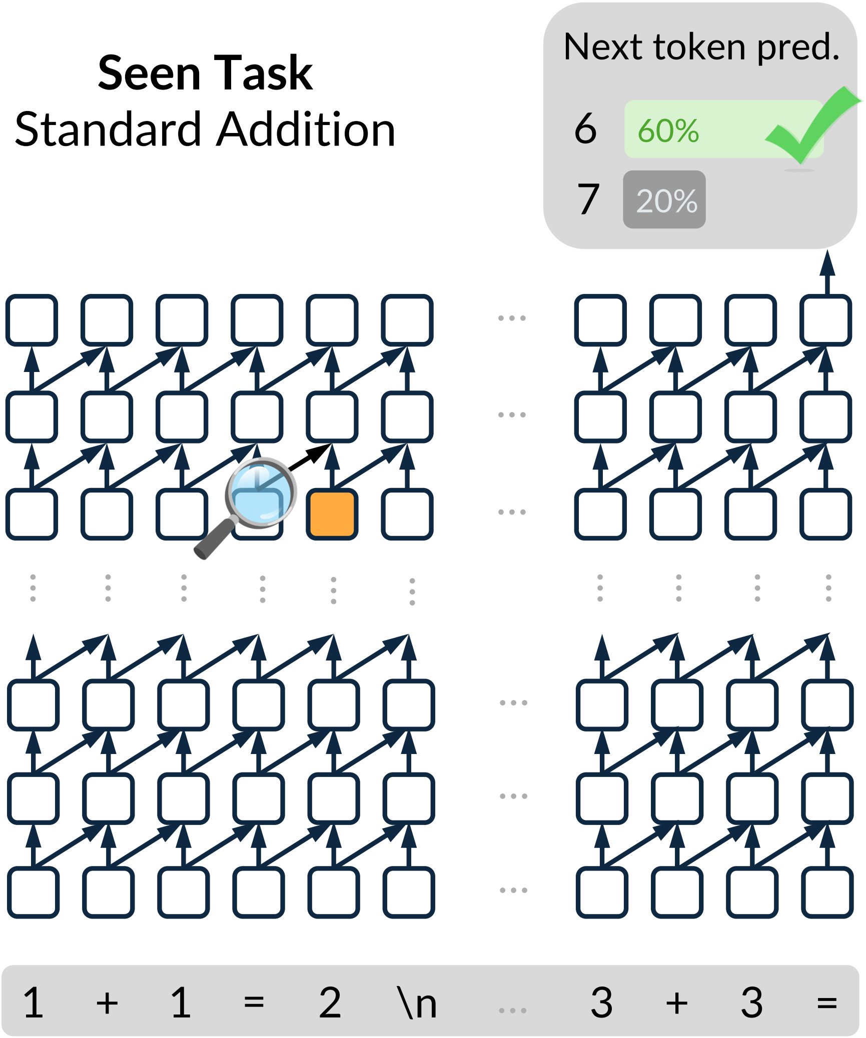

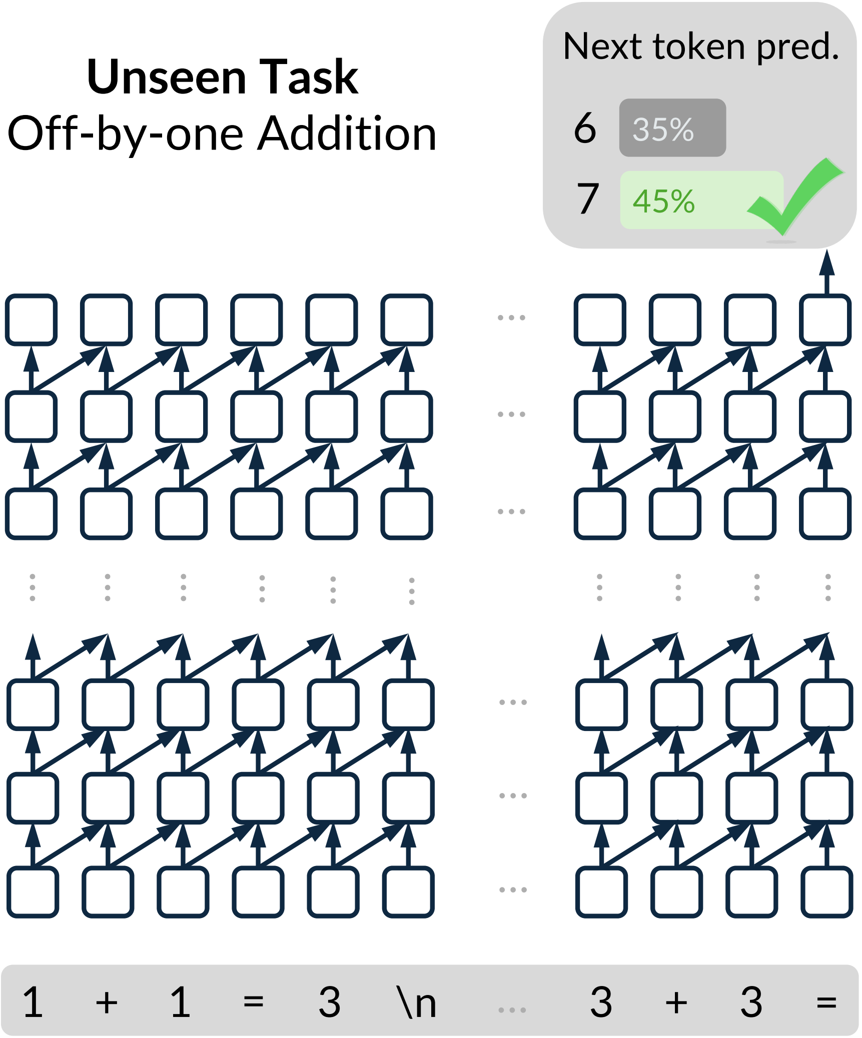

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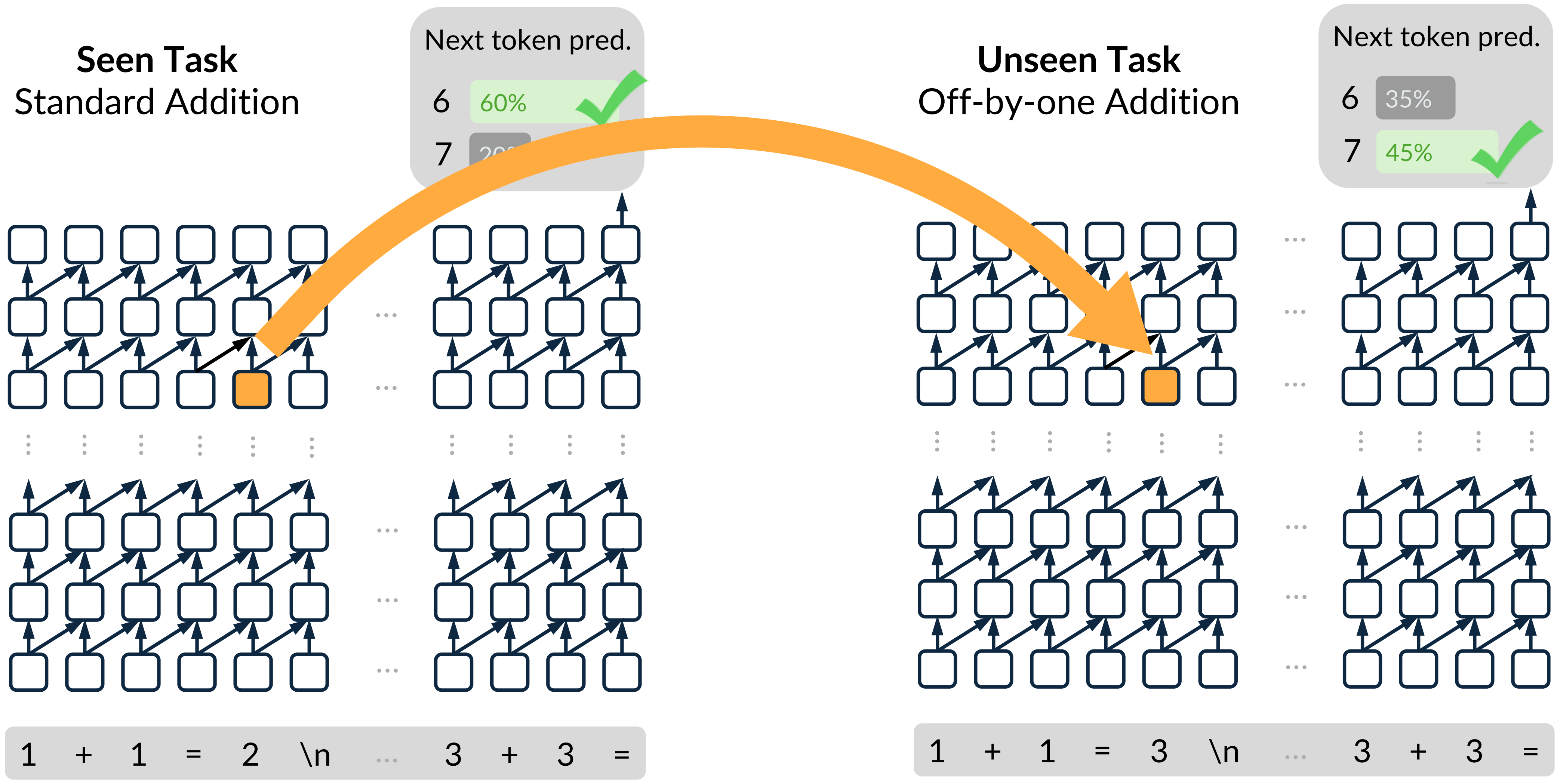

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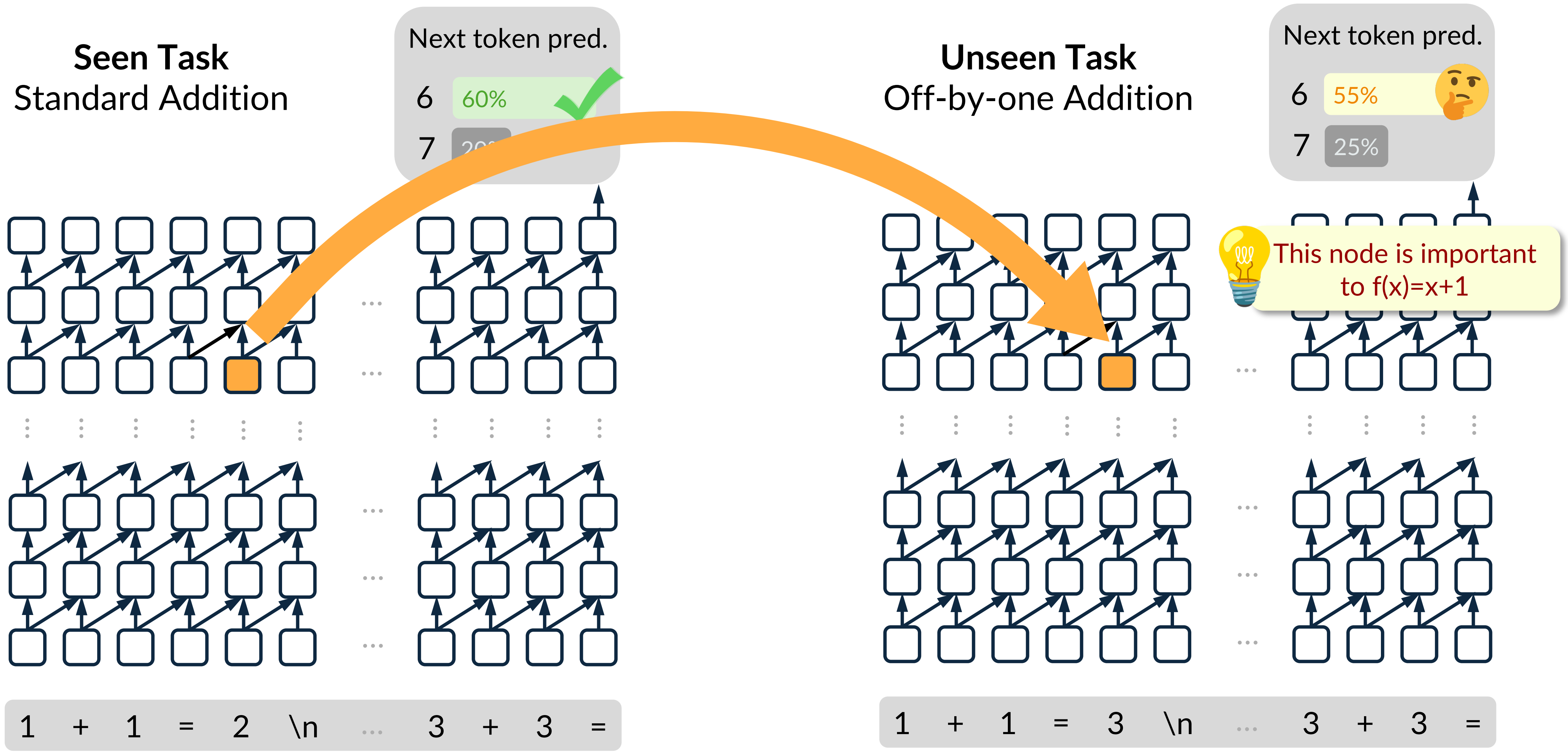
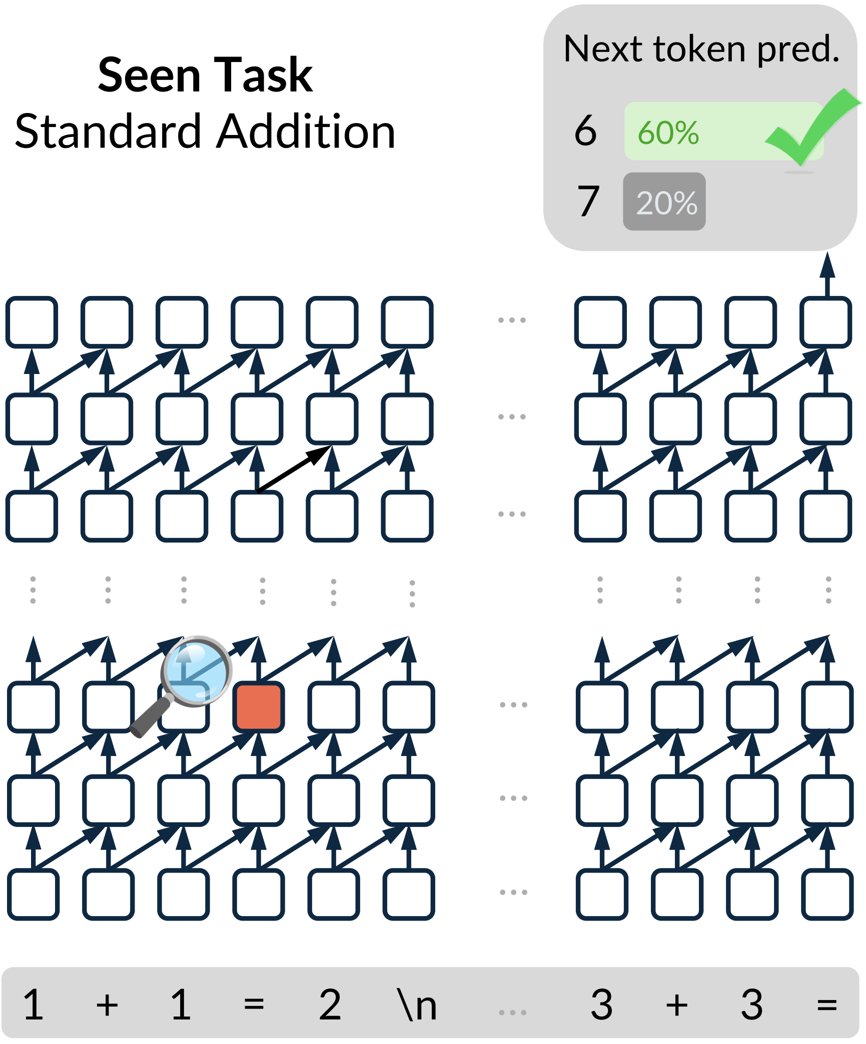
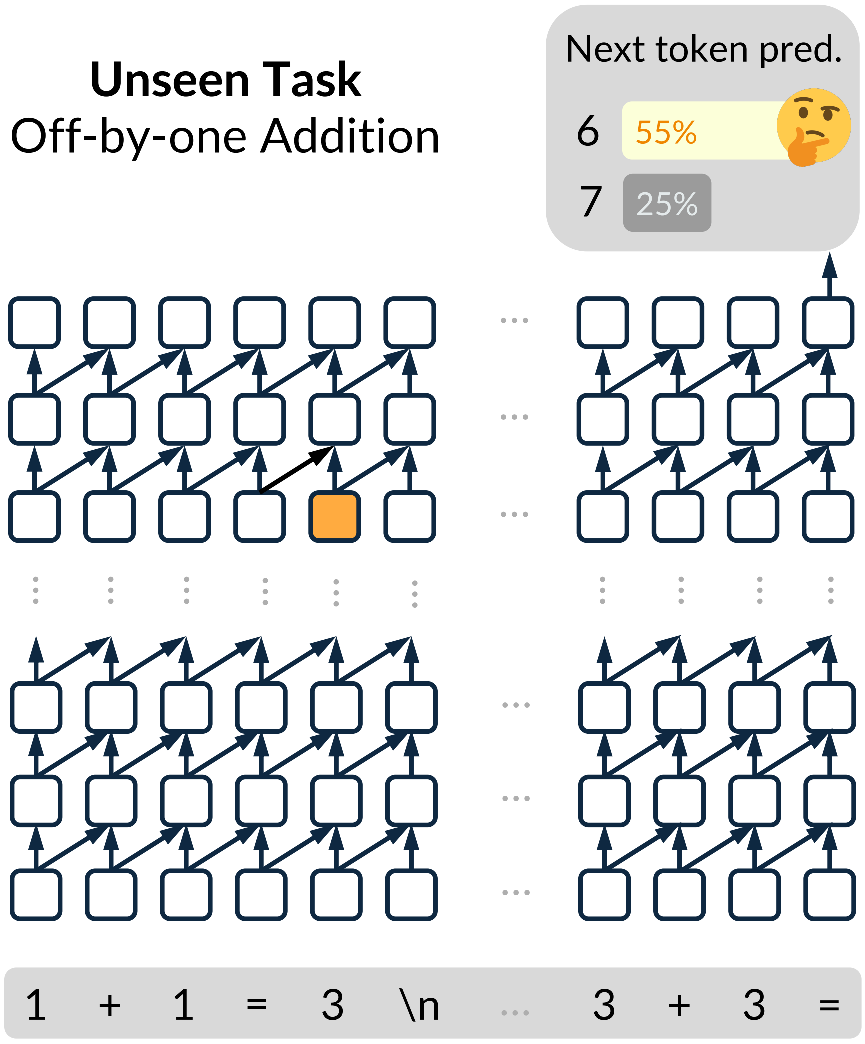


Interpreting Model Internals with Patching





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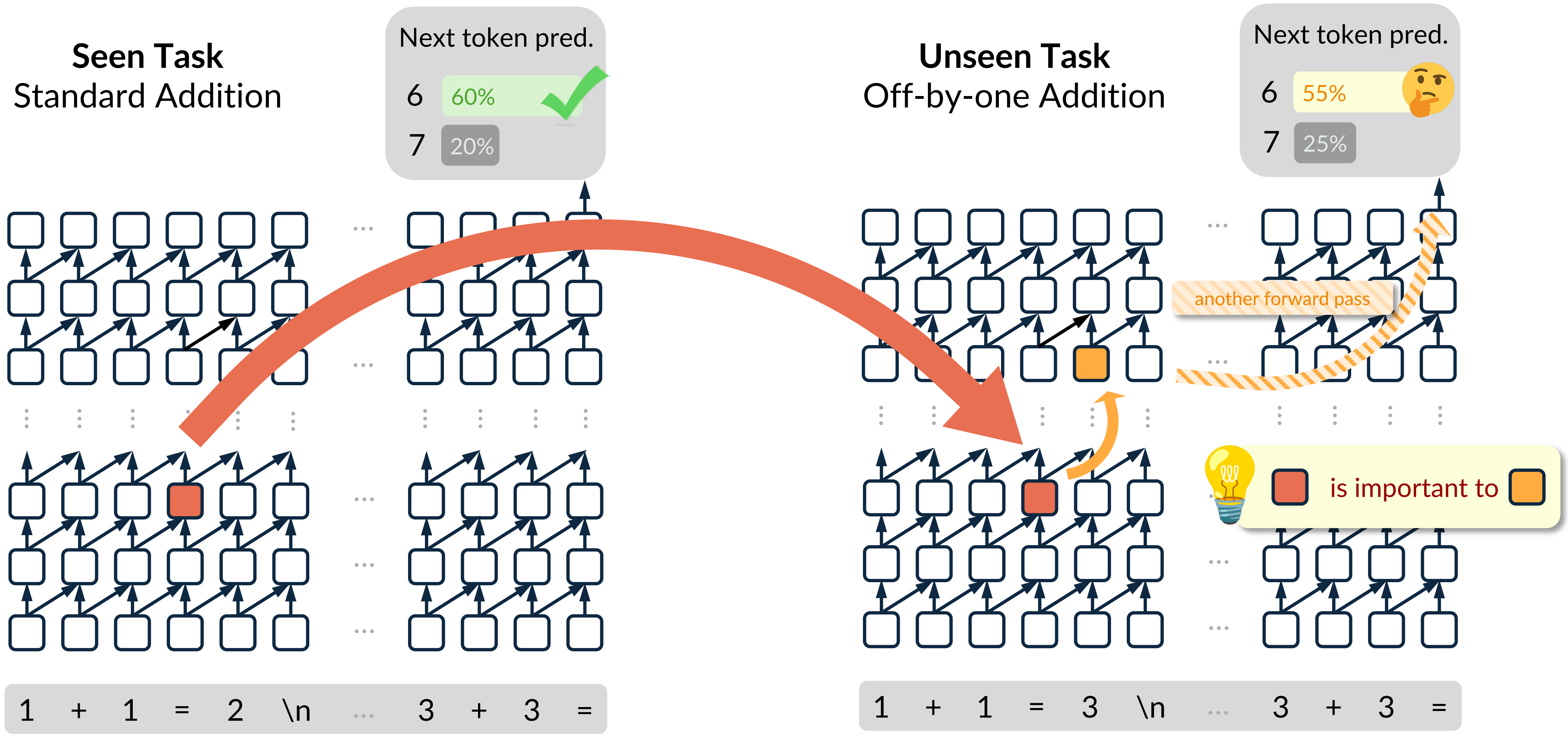

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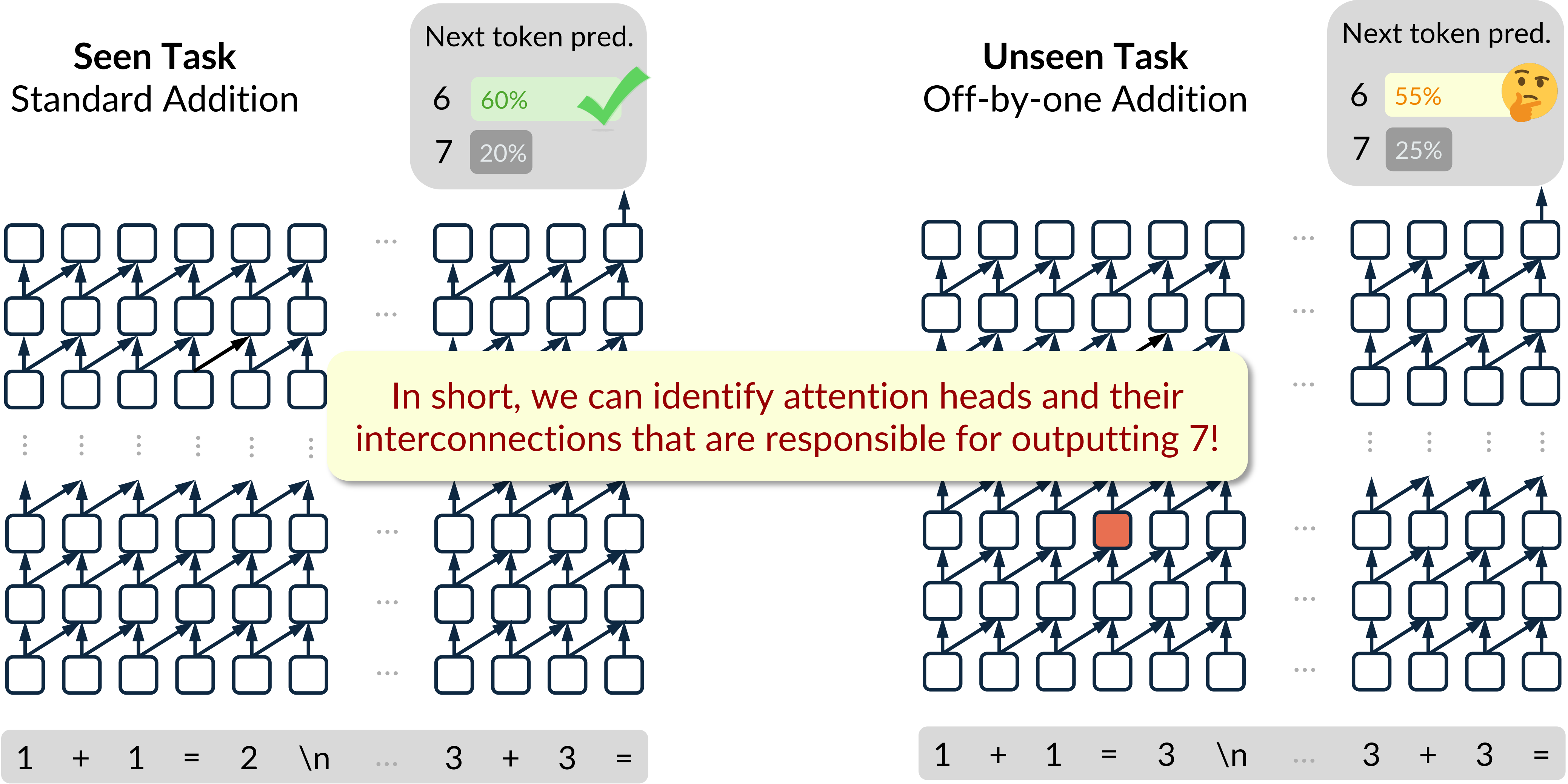
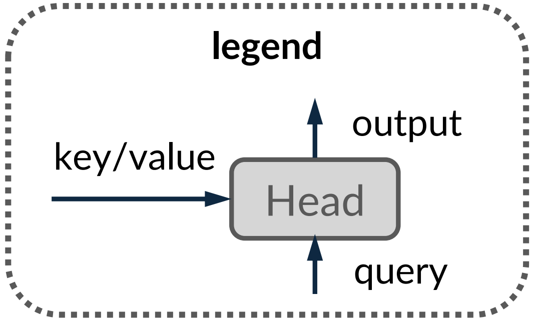


In short, we can identify attention heads and their interconnections that are responsible for outputting 7!

Patching with Contrast Tasks

Next token pred.

6	35%
7	45% ✓

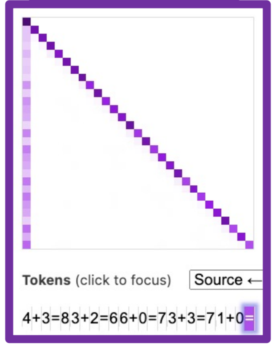


Early layers

1	+	1	=	3	\n	...	3	+	3	=

Patching with Contrast Tasks

Group 1



Next token pred.

6 35%

7 45%

Note: H41.4 means head 4 in layer 41.

H41.4, H41.5, ...

legend

key/value

output

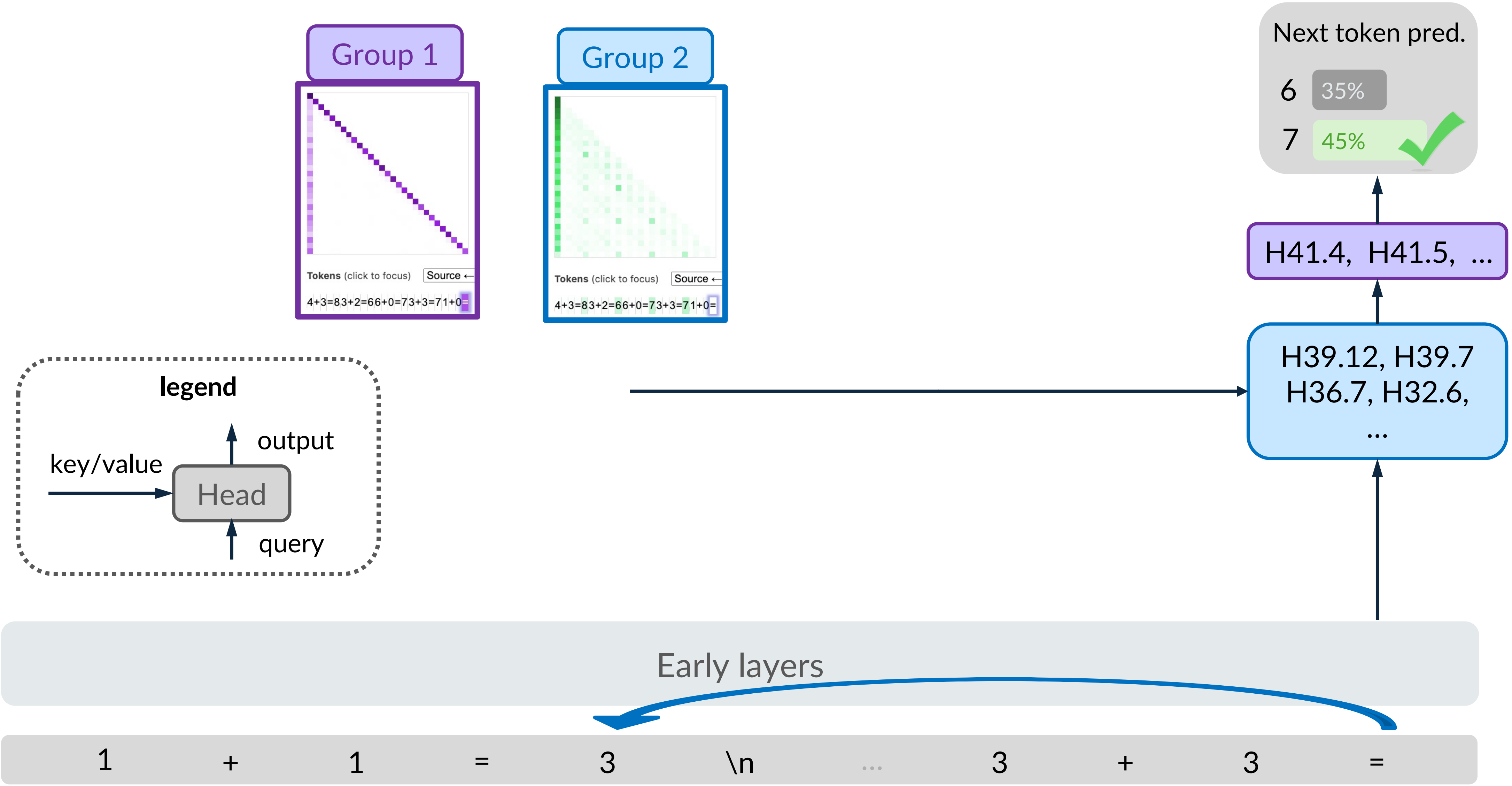
Head

query

Early layers

1	+	1	=	3	\n	...	3	+	3	=

Patching with Contrast Tasks



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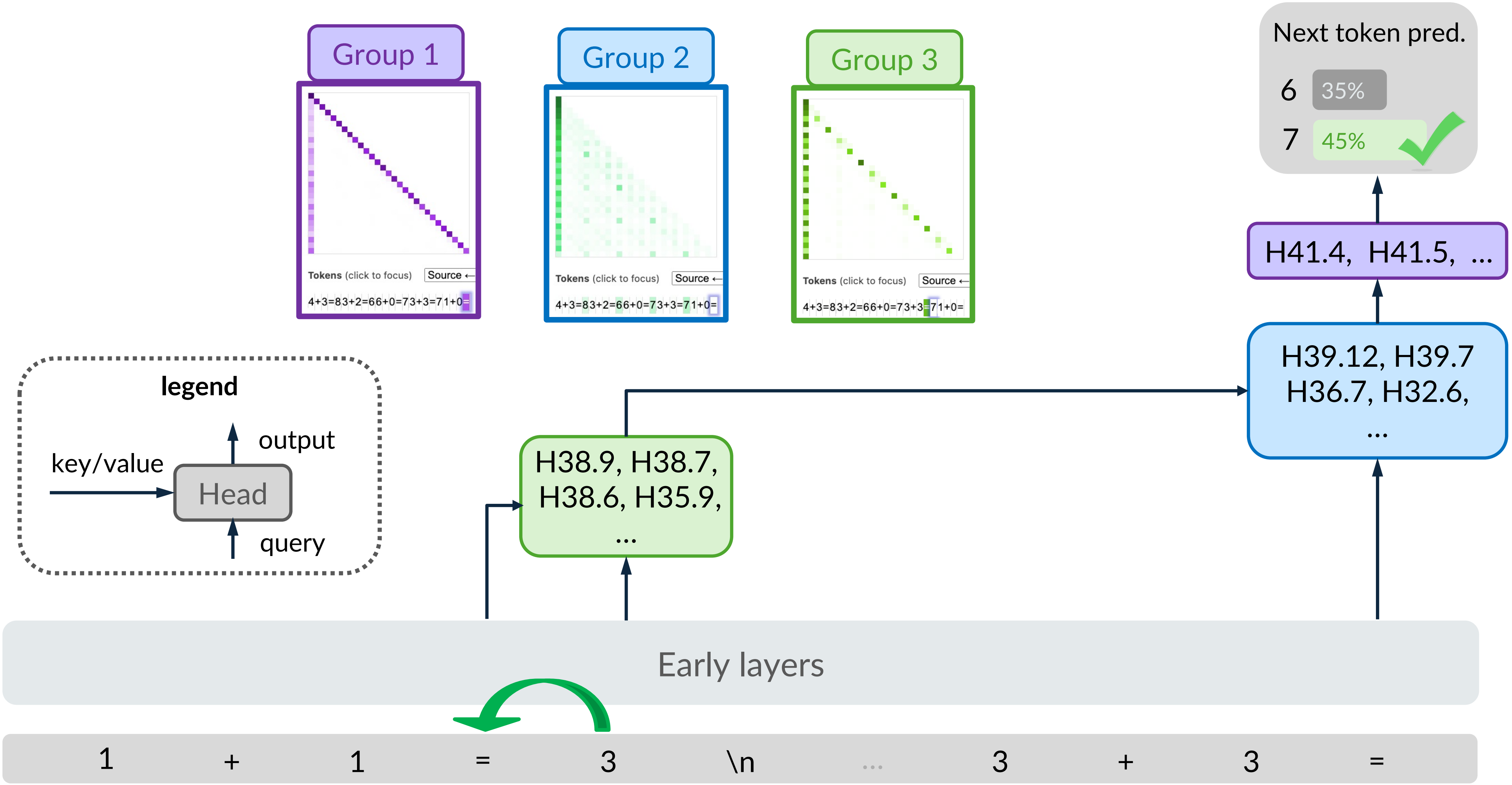
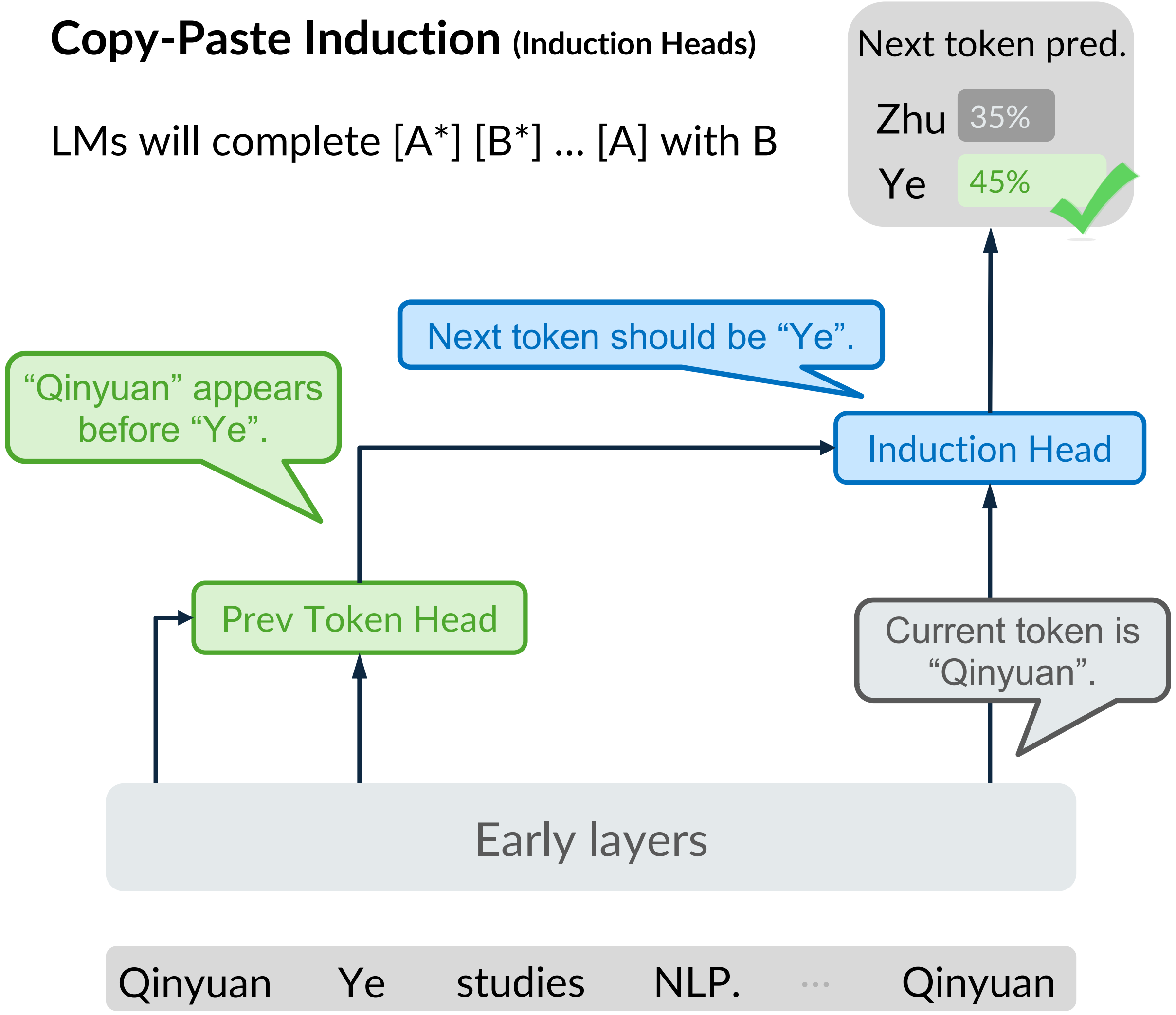


Revisiting Induction Heads

Copy-Paste Induction (Induction Heads)

LMs will complete [A*] [B*] … [A] with B



In-context Learning and Induction Heads (Olsson et al., 2022)

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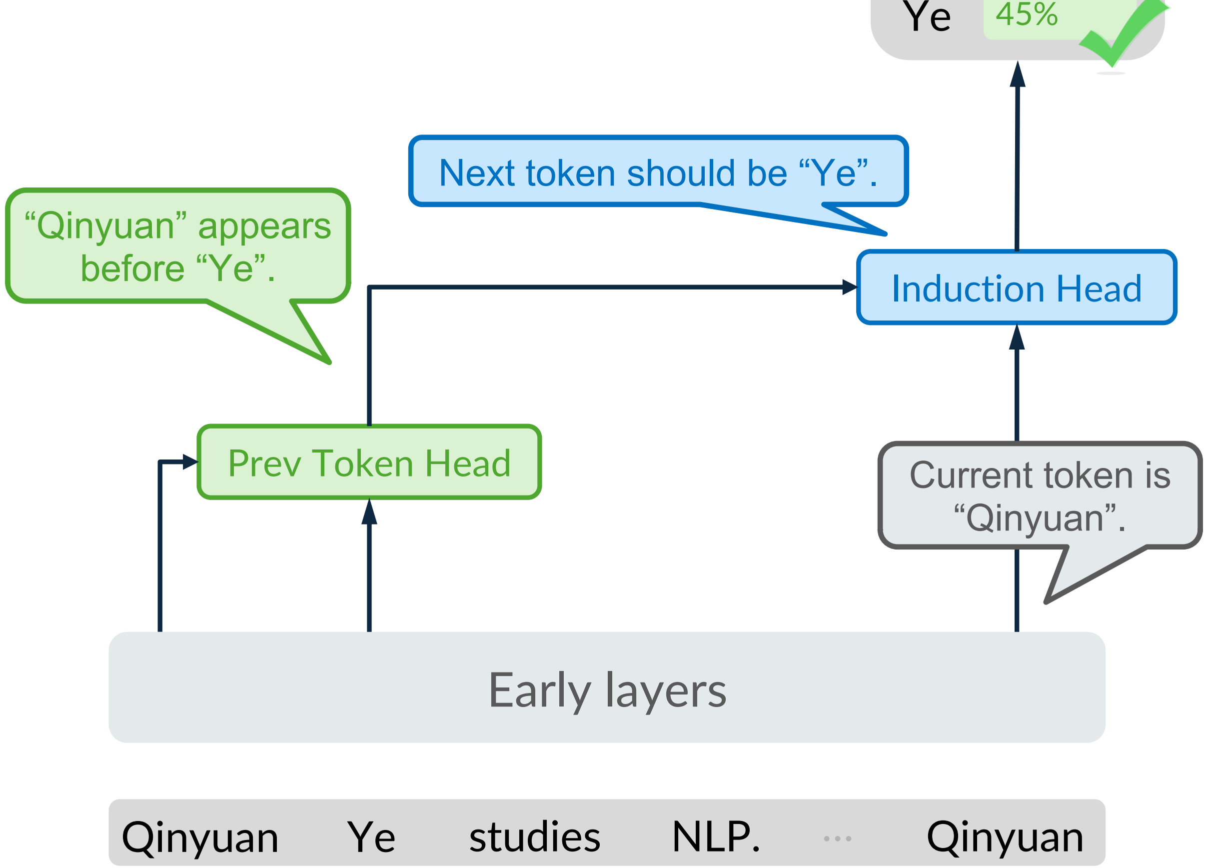
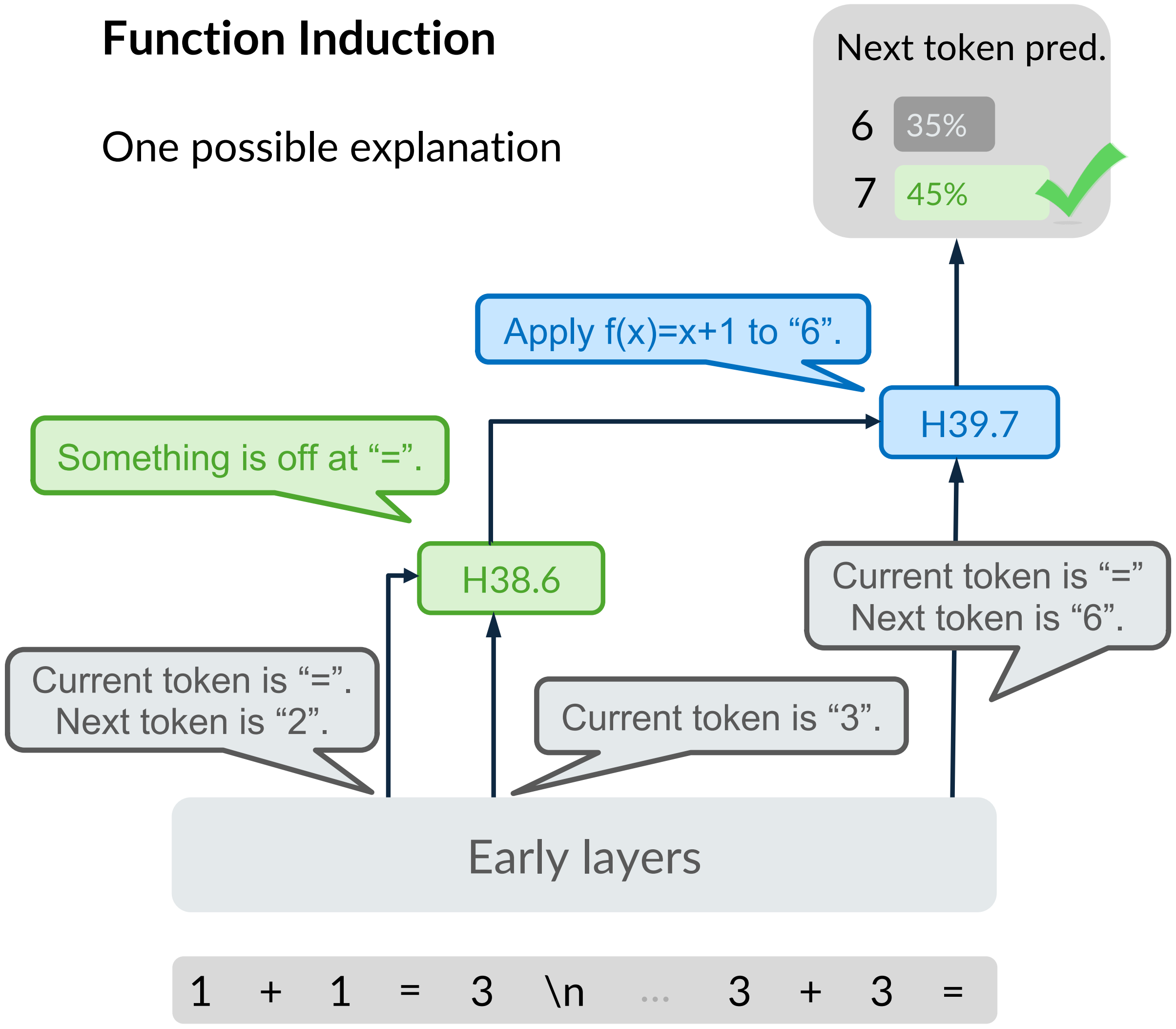


In-context Learning and Induction Heads (Olsson et al., 2022)

Function Induction

One possible explanation



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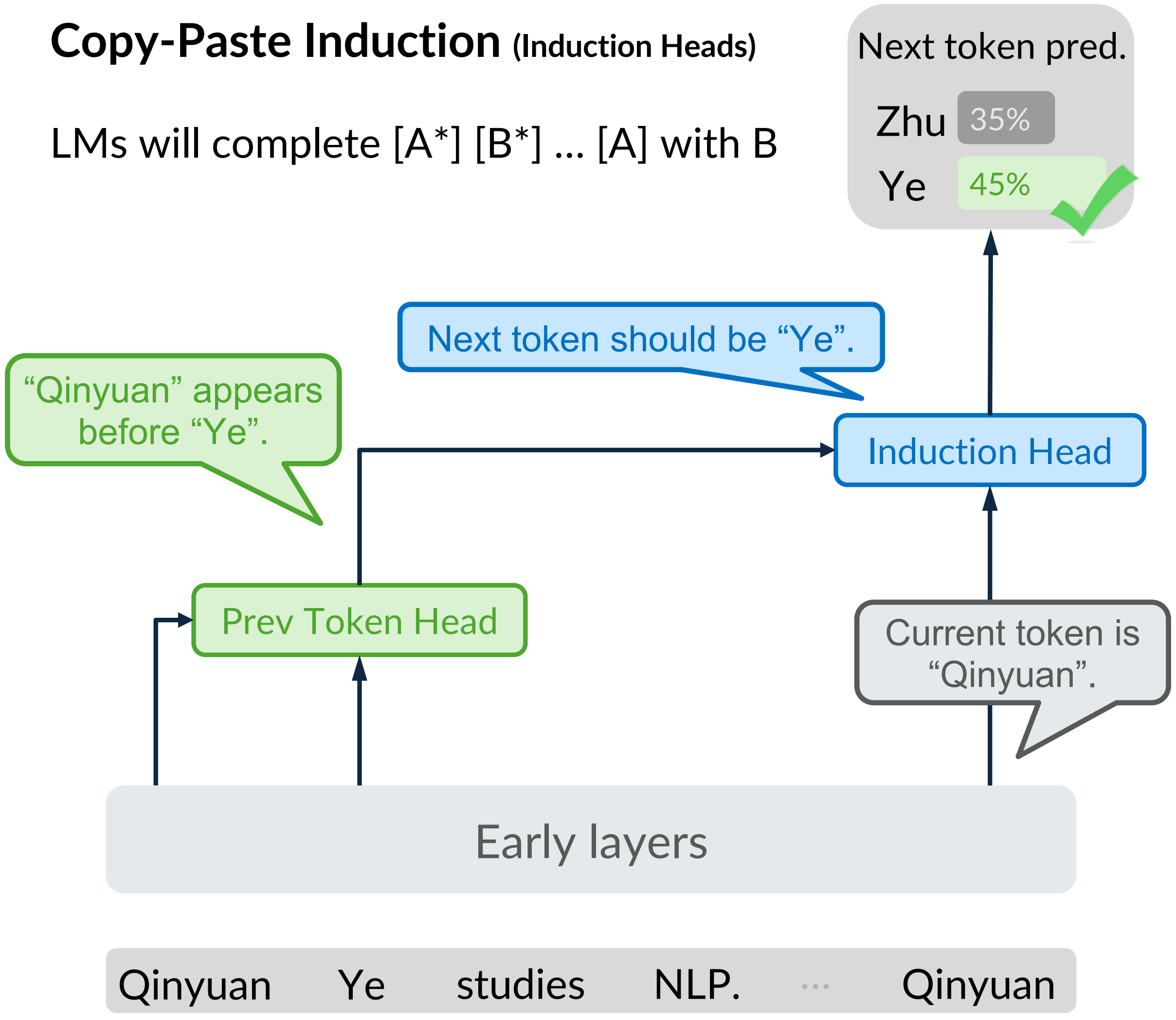

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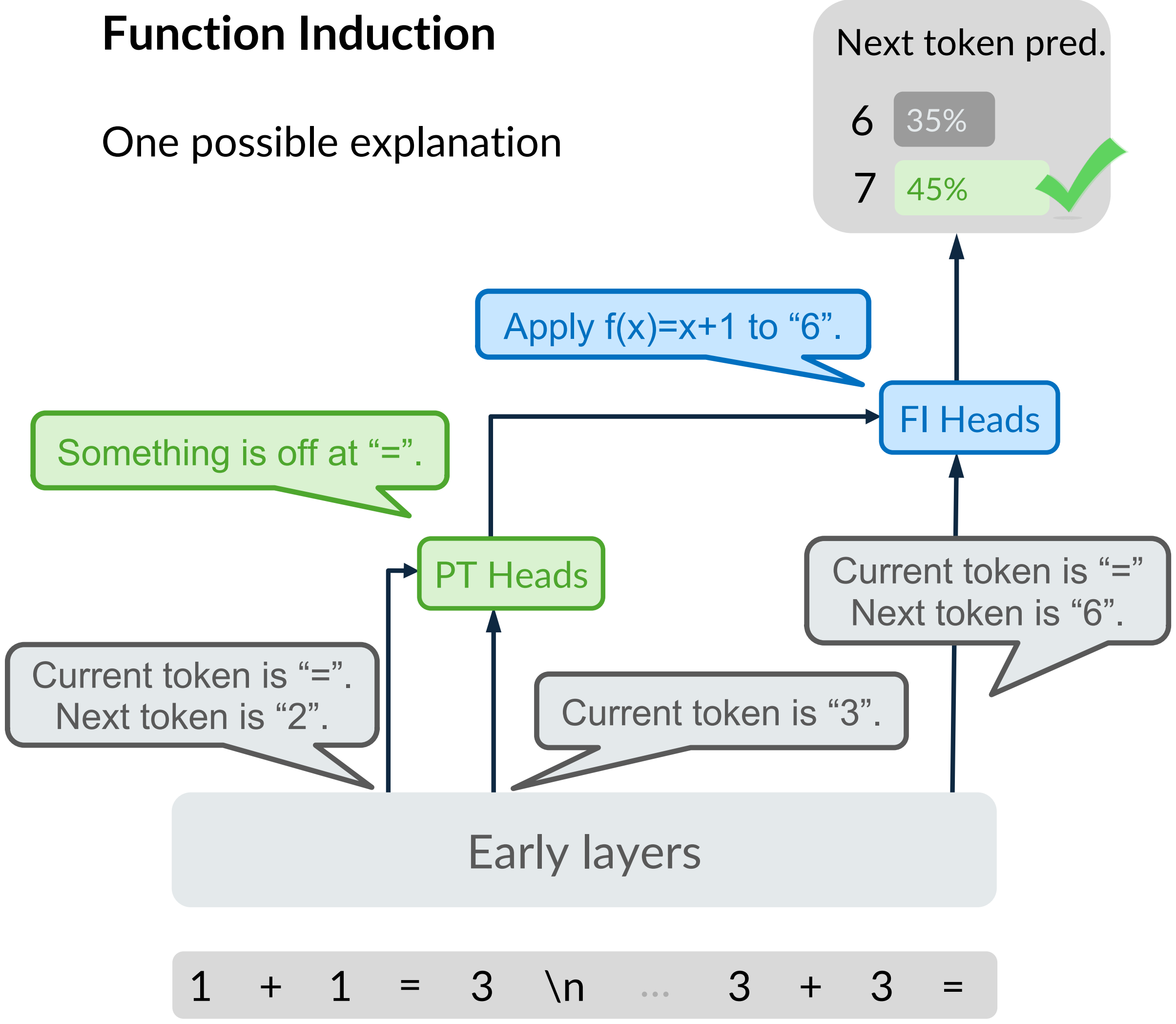
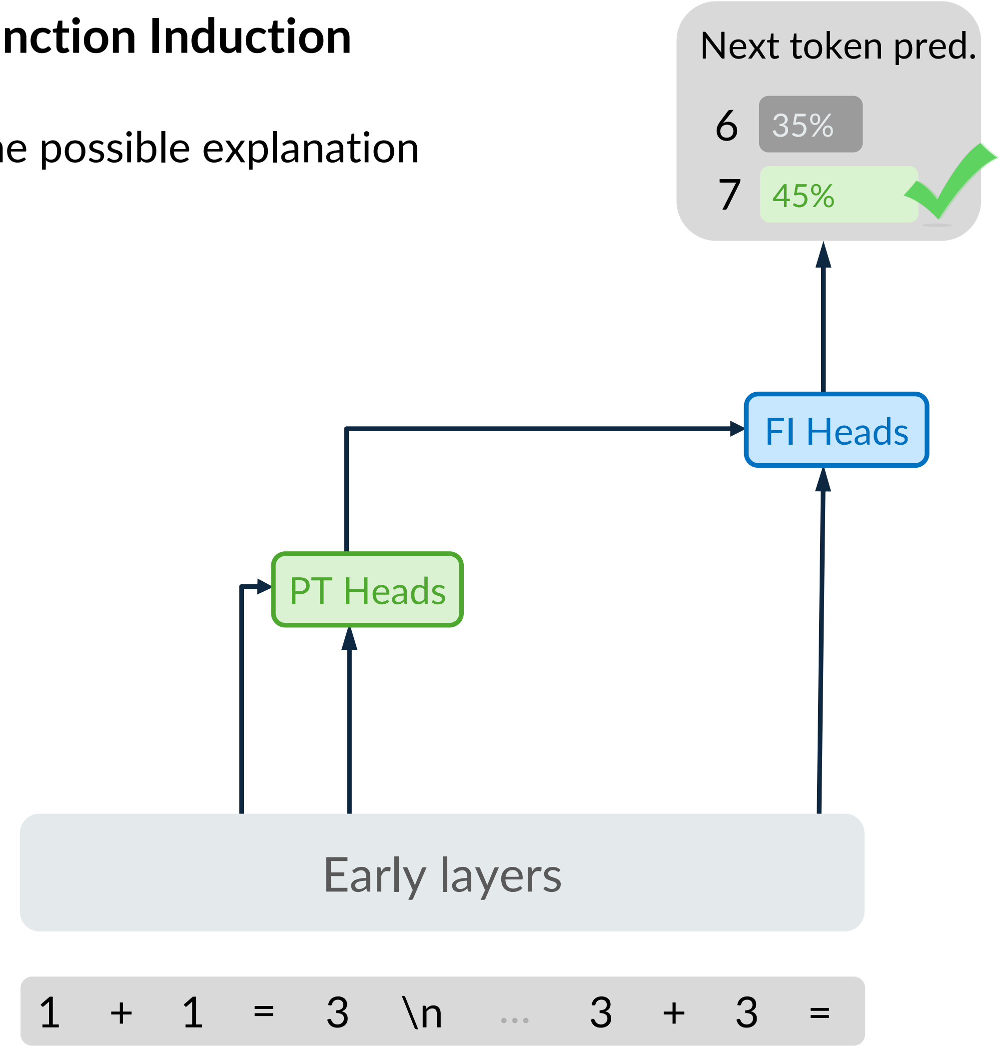


Finding 1: Function Induction Mechanism

- LMs *may be* implementing a complex **function induction** mechanism.
  - Generalizes the findings in Olsson et al., 2022;
  - Elevates it from the token level to the function level.

Function Induction

One possible explanation



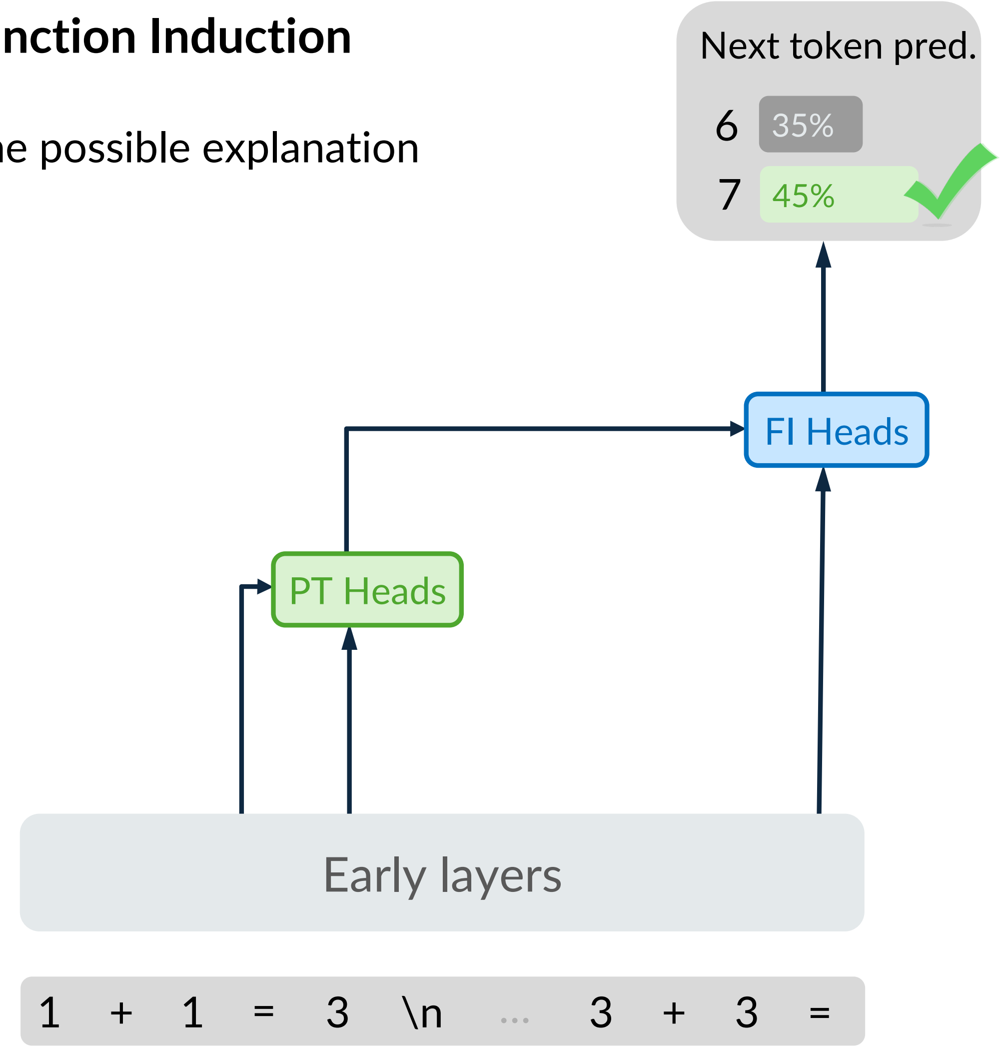
Finding 1: Function Induction Mechanism

- LMs *may be* implementing a complex **function induction** mechanism.
  - Generalizes the findings in Olsson et al., 2022;
  - Elevates it from the token level to the function level.

- ➔ **More questions**
  - Are **these heads** really writing out *f(x)=x+1*?
  - If *f(x)=x+1* is emitted 9 times via 9 heads, why is it not interpreted as "+9" by the model?

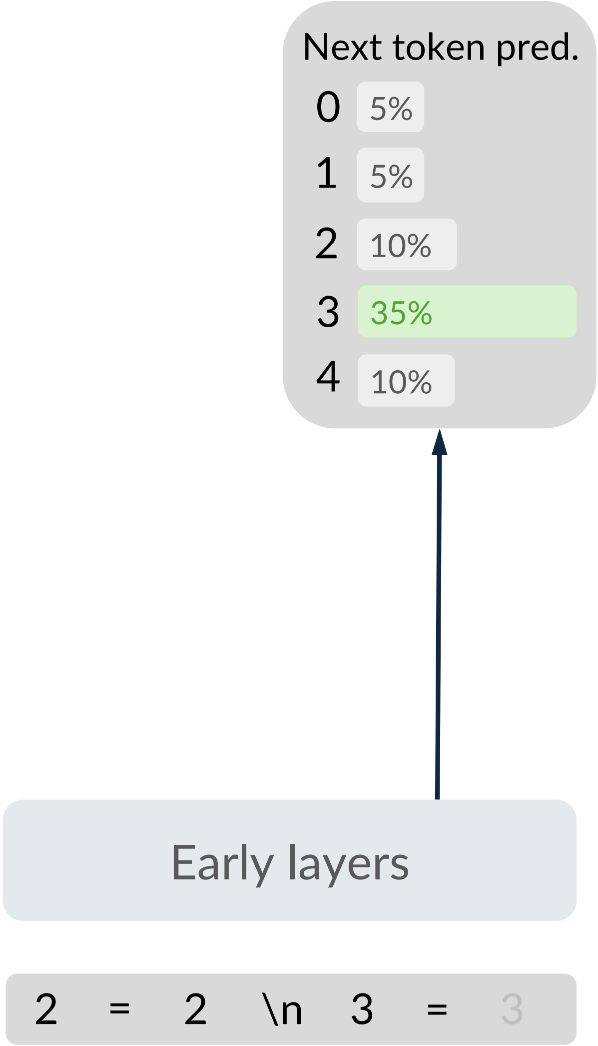
Function Induction

One possible explanation



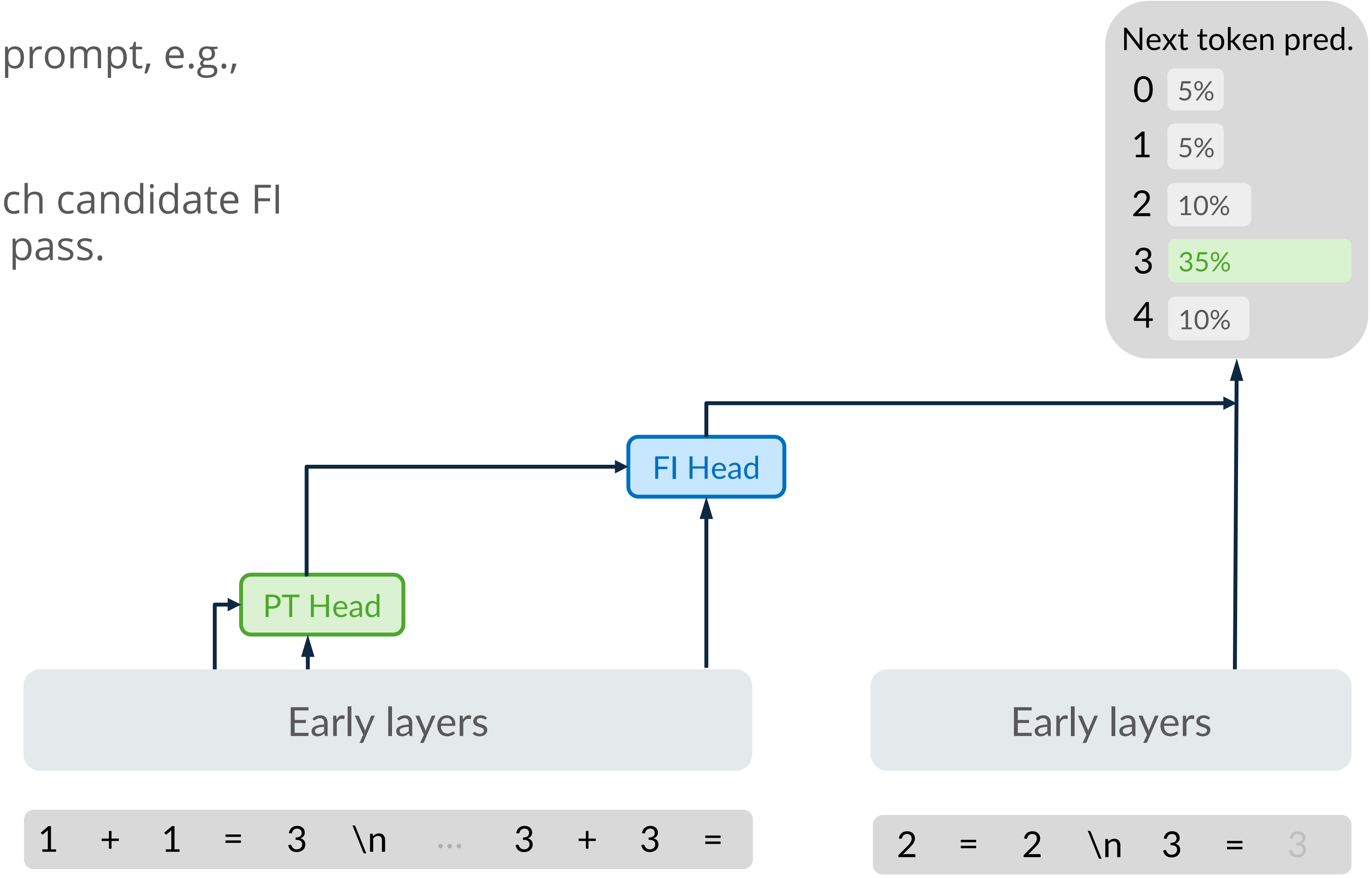
Investigating Each Candidate FI Head

- We run the LM on a naive prompt, e.g.,
  2=2, 3=?



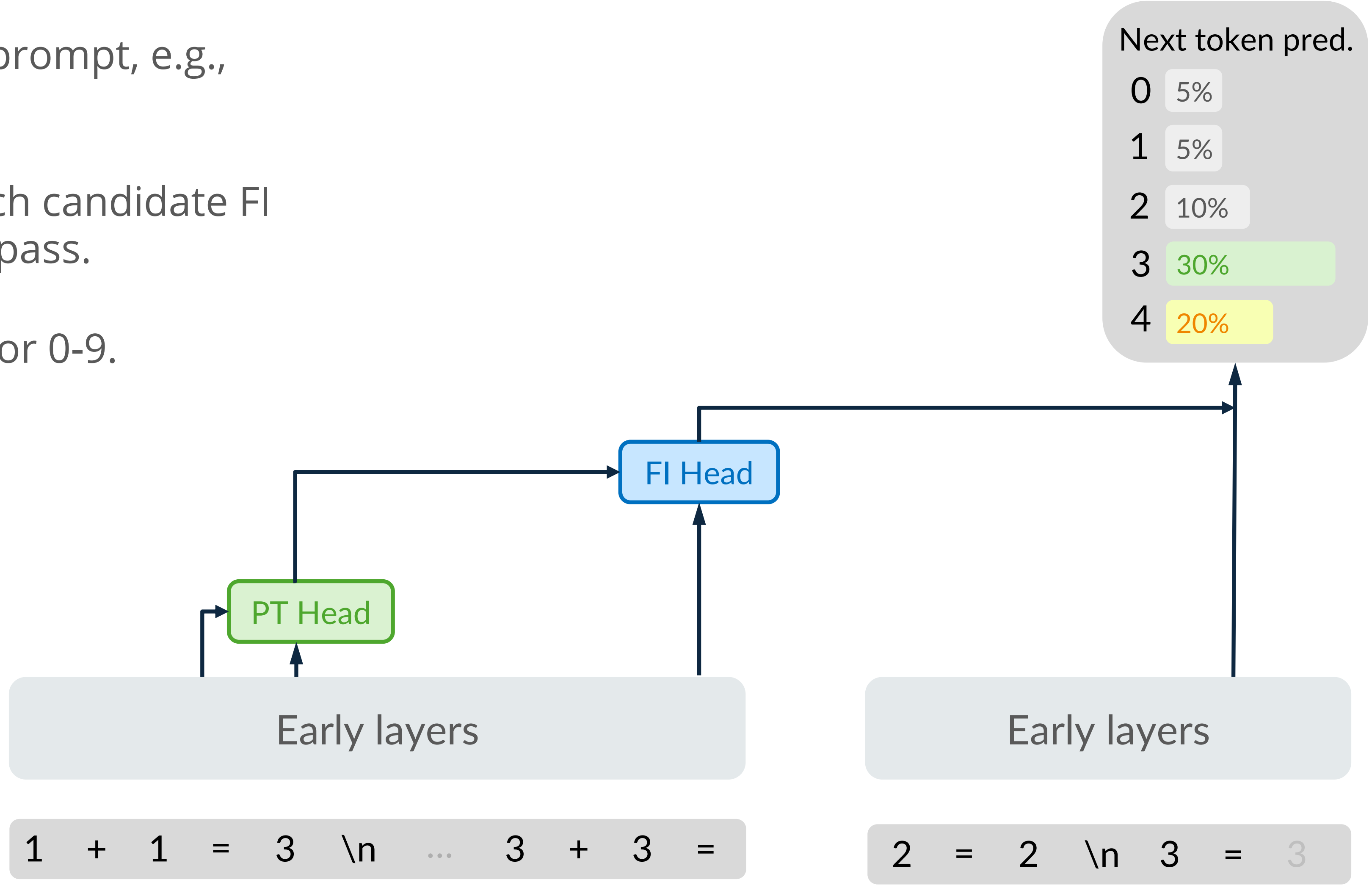
Investigating Each Candidate FI Head

- We run the LM on a naive prompt, e.g., 2=2, 3=?
- We patch the output of each candidate FI head to the naive forward pass.



Investigating Each Candidate FI Head

- We run the LM on a naive prompt, e.g., 2=2, 3=?
- We patch the output of each candidate FI head to the naive forward pass.
- We track the logit change for 0-9.



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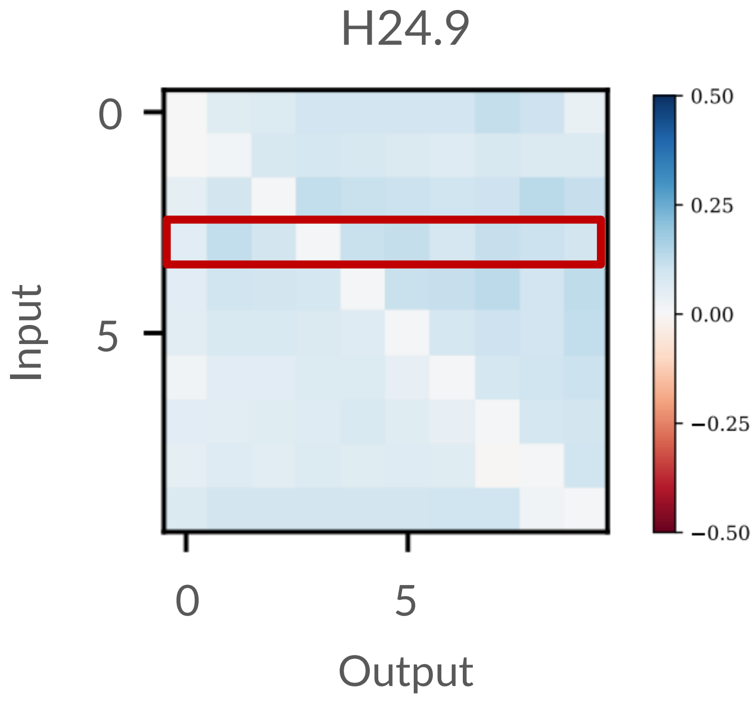
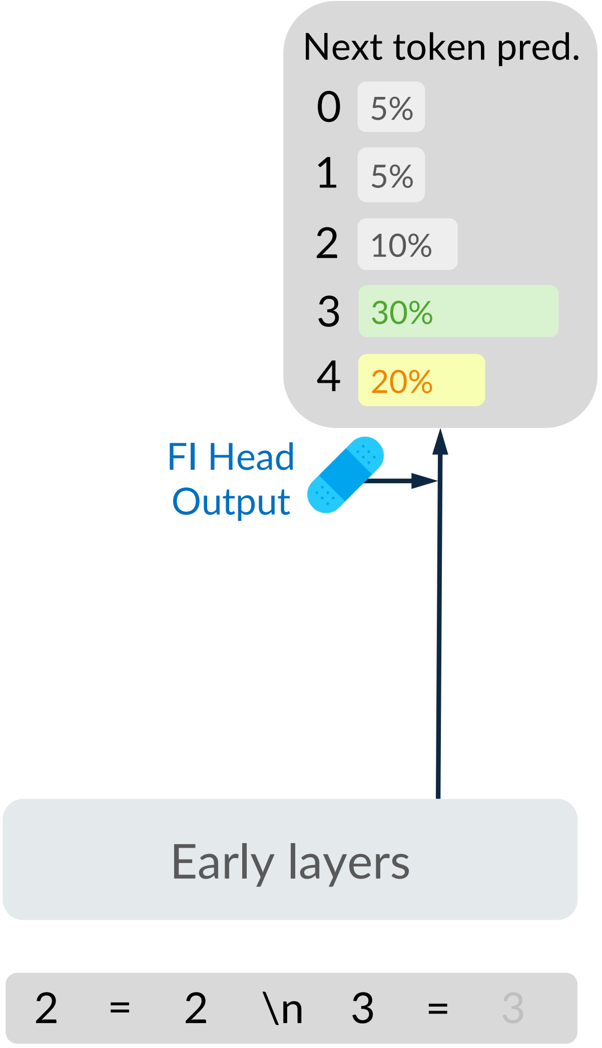
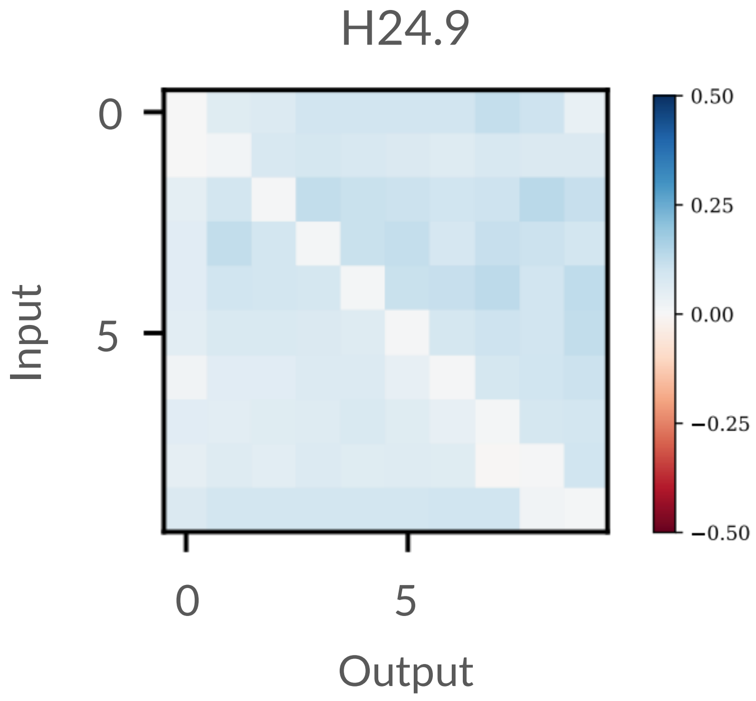


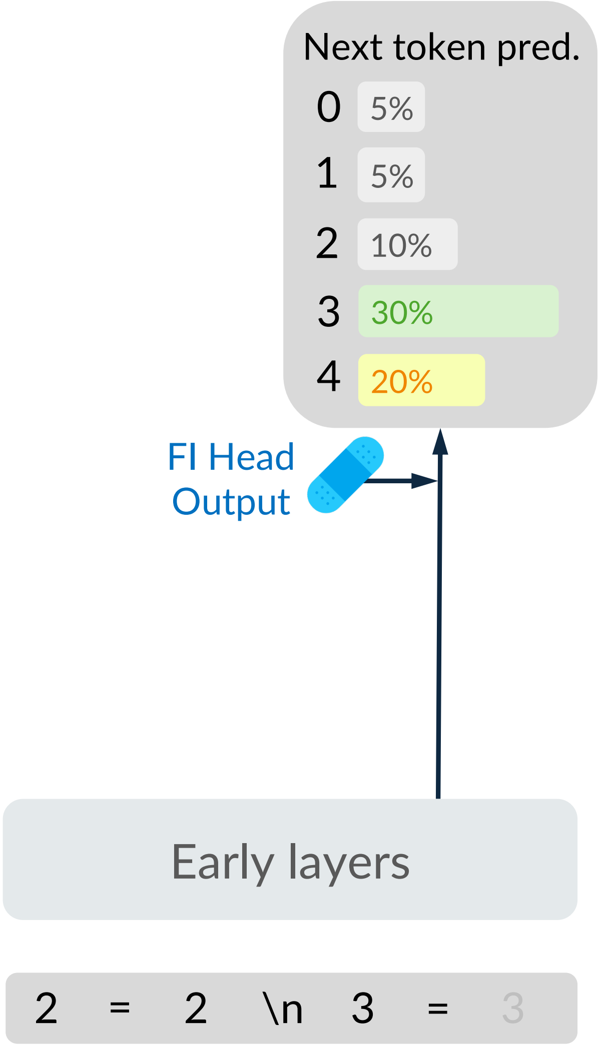
When the input is 3,
All tokens other than 3 are promoted. 3 is suppressed.



Investigating Each Candidate FI Head

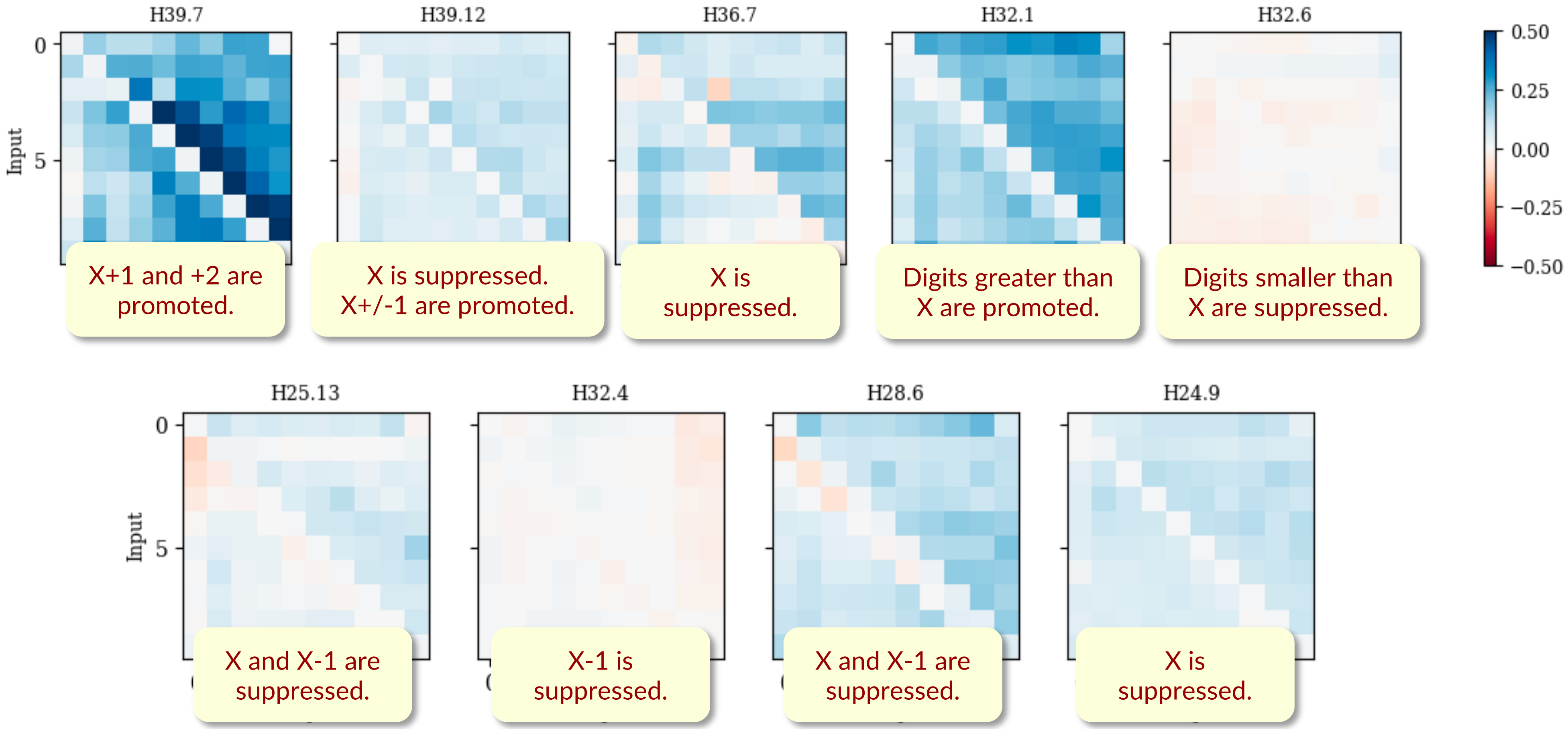


When input is X,
X is suppressed.



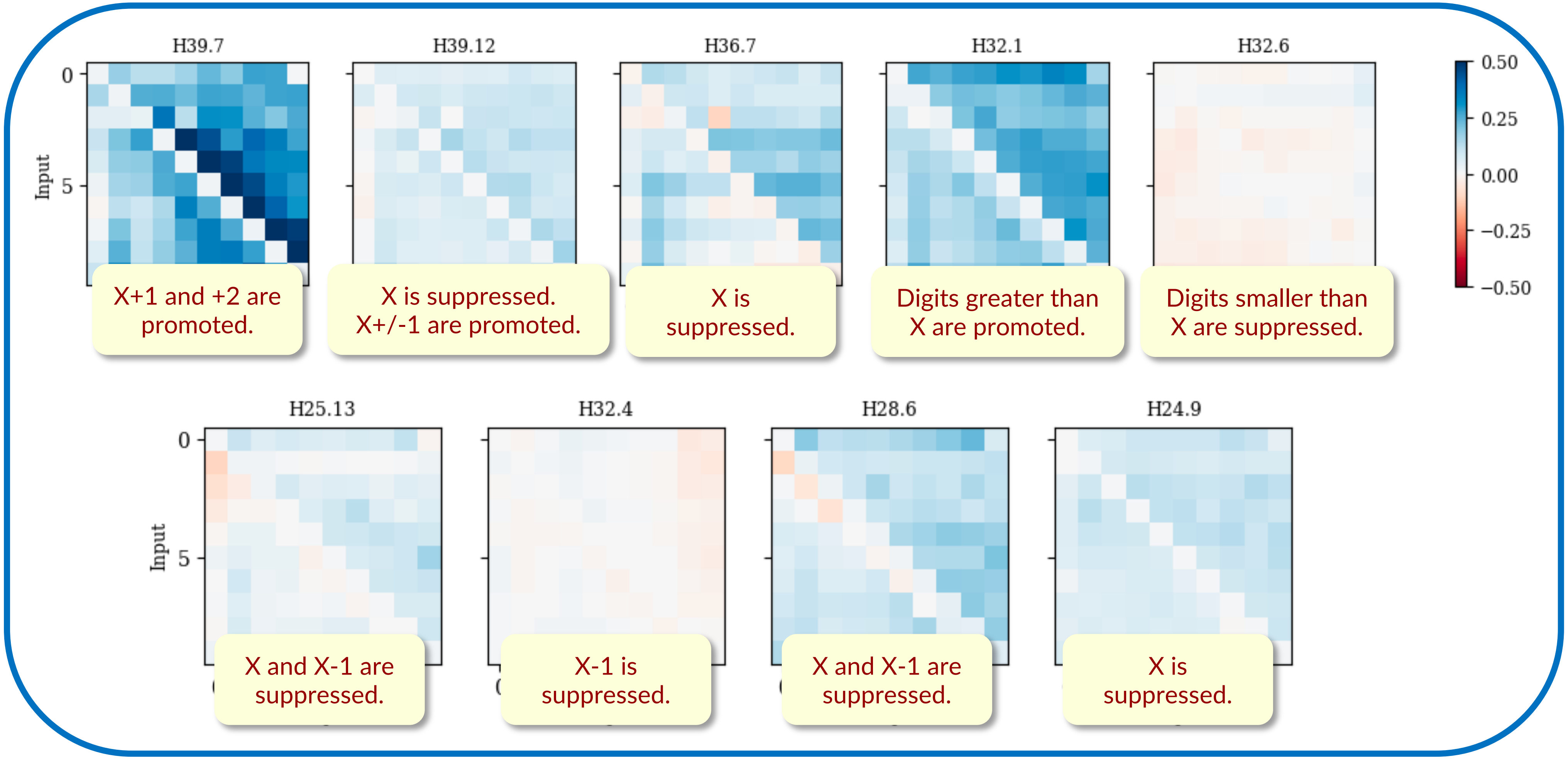
Investigating Each Candidate FI Head

When the input is X

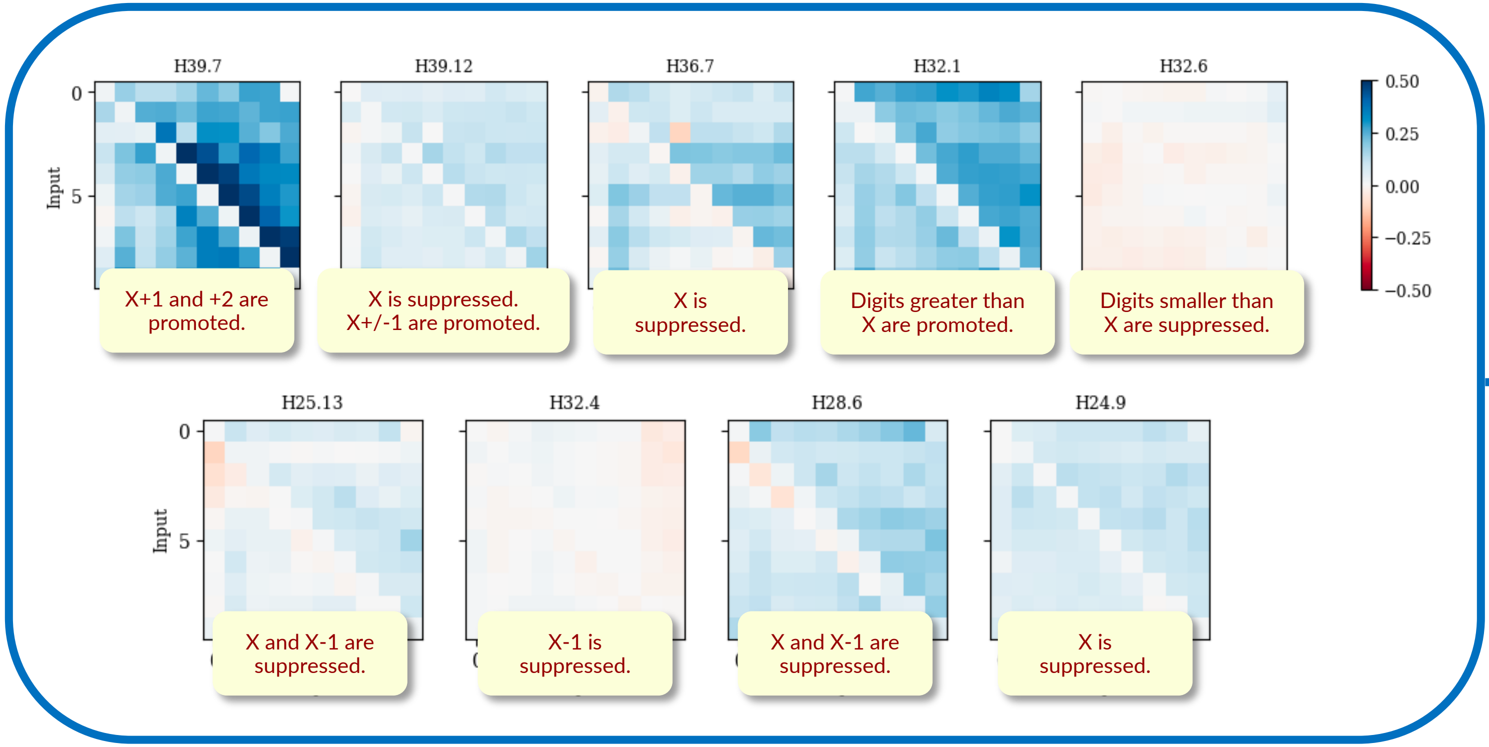


Investigating Each Candidate FI Head

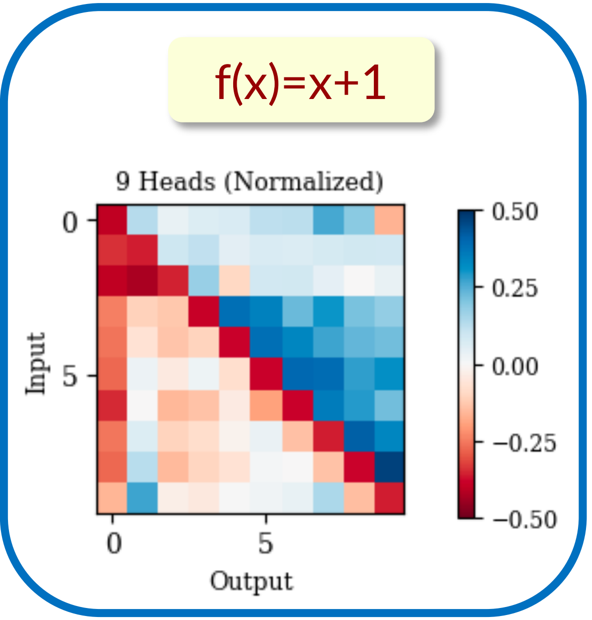
When the input is X



Finding 2: FI Heads Work Collaboratively!

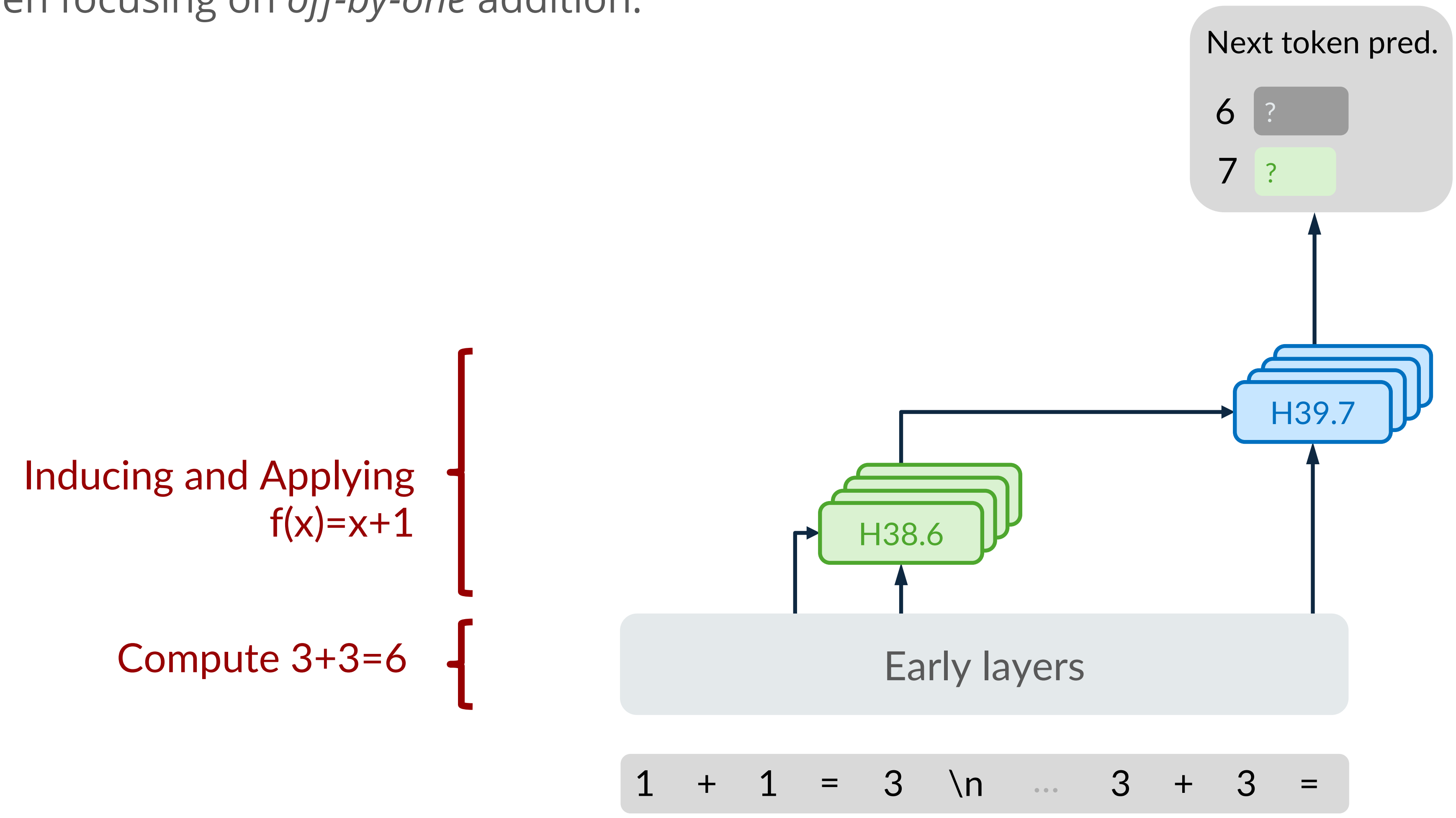


aggregate



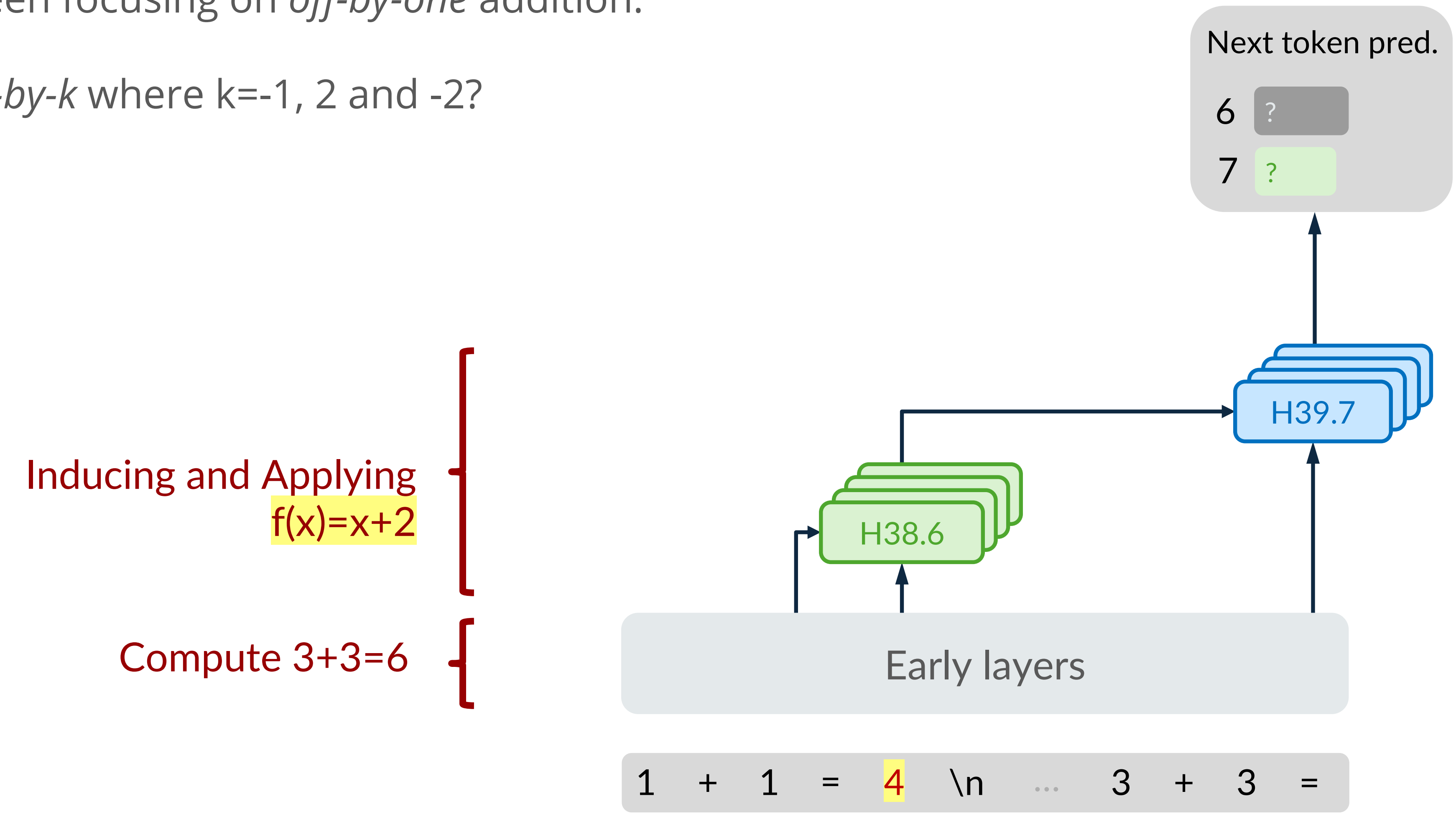
From Off-by-one to Off-by-k Addition

- So far, we've been focusing on *off-by-one* addition.



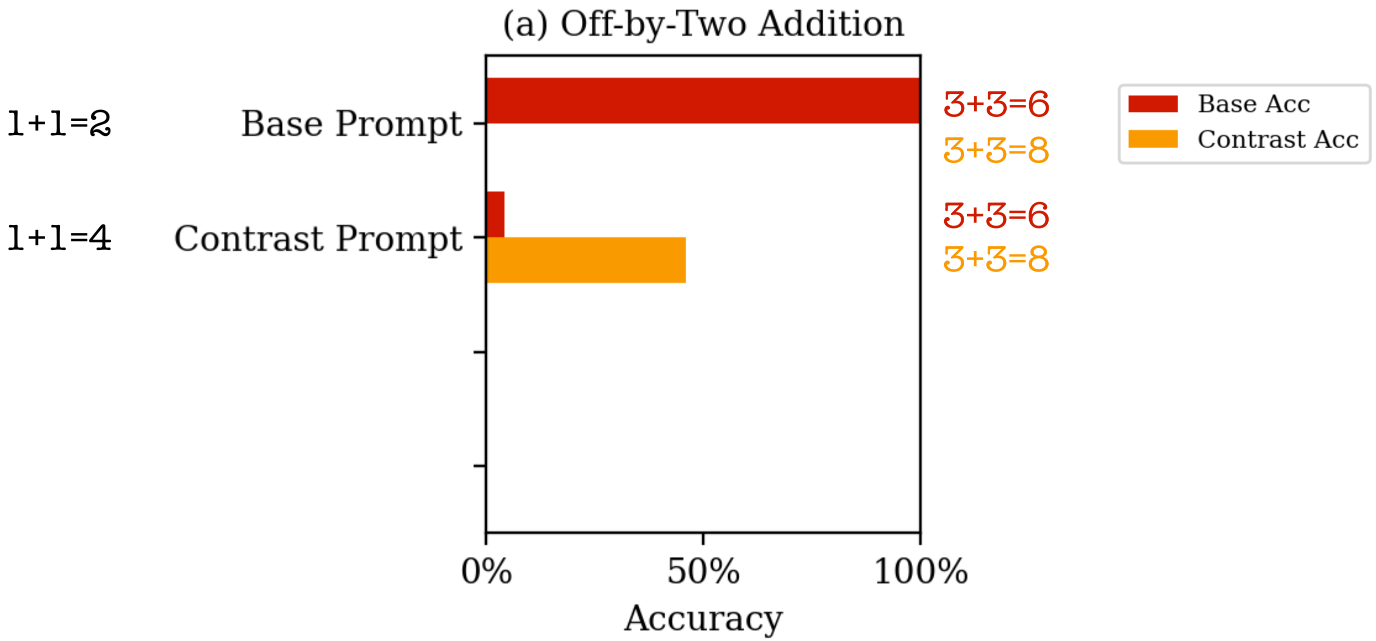
From Off-by-one to Off-by-k Addition

- So far, we've been focusing on *off-by-one* addition.
- What about *off-by-k* where k=-1, 2 and -2?



From Off-by-one to Off-by-k Addition

- We investigate this with head ablation experiments.



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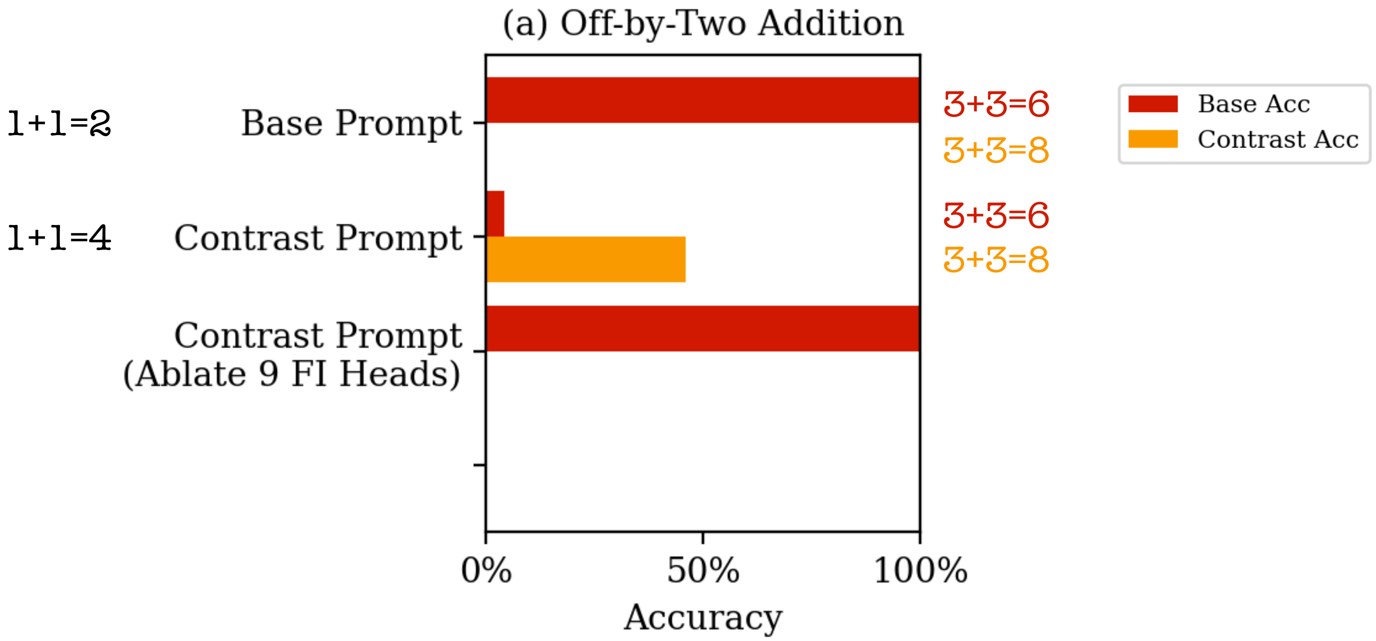

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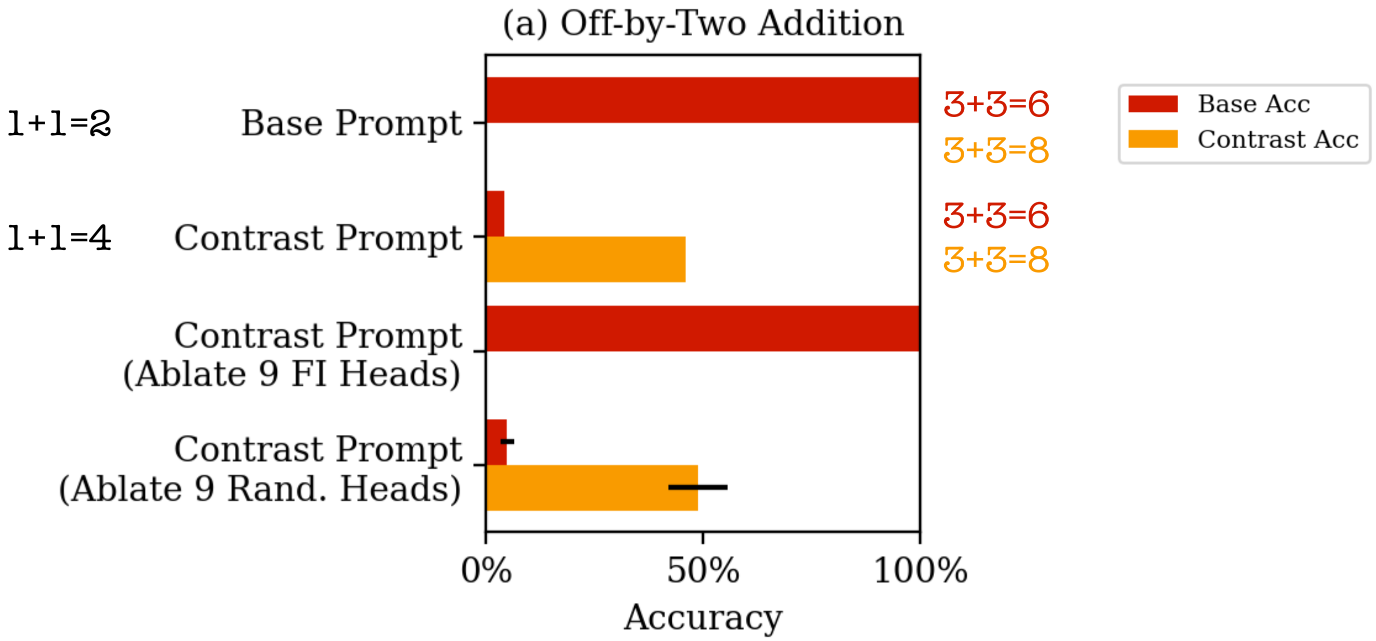
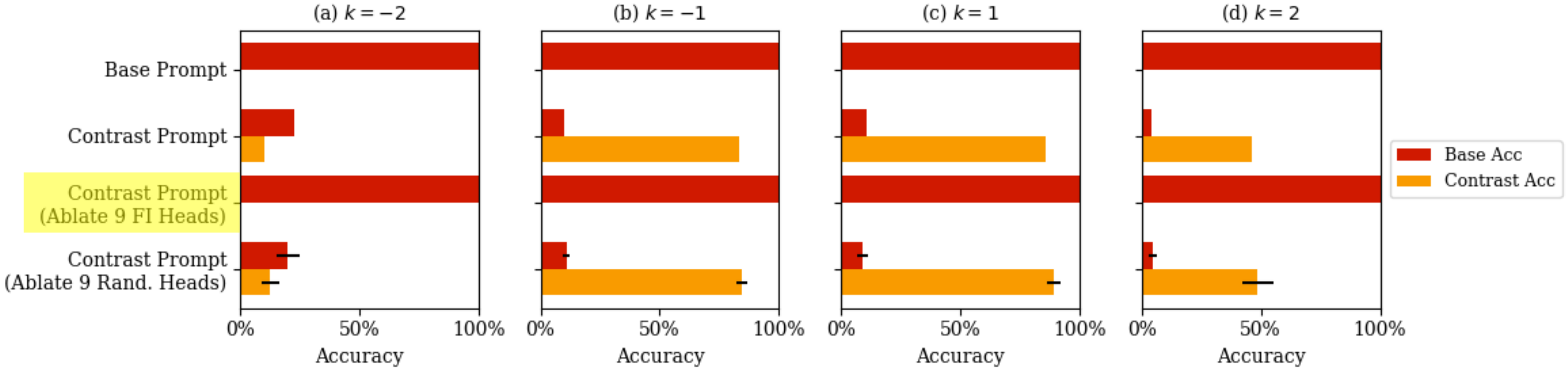


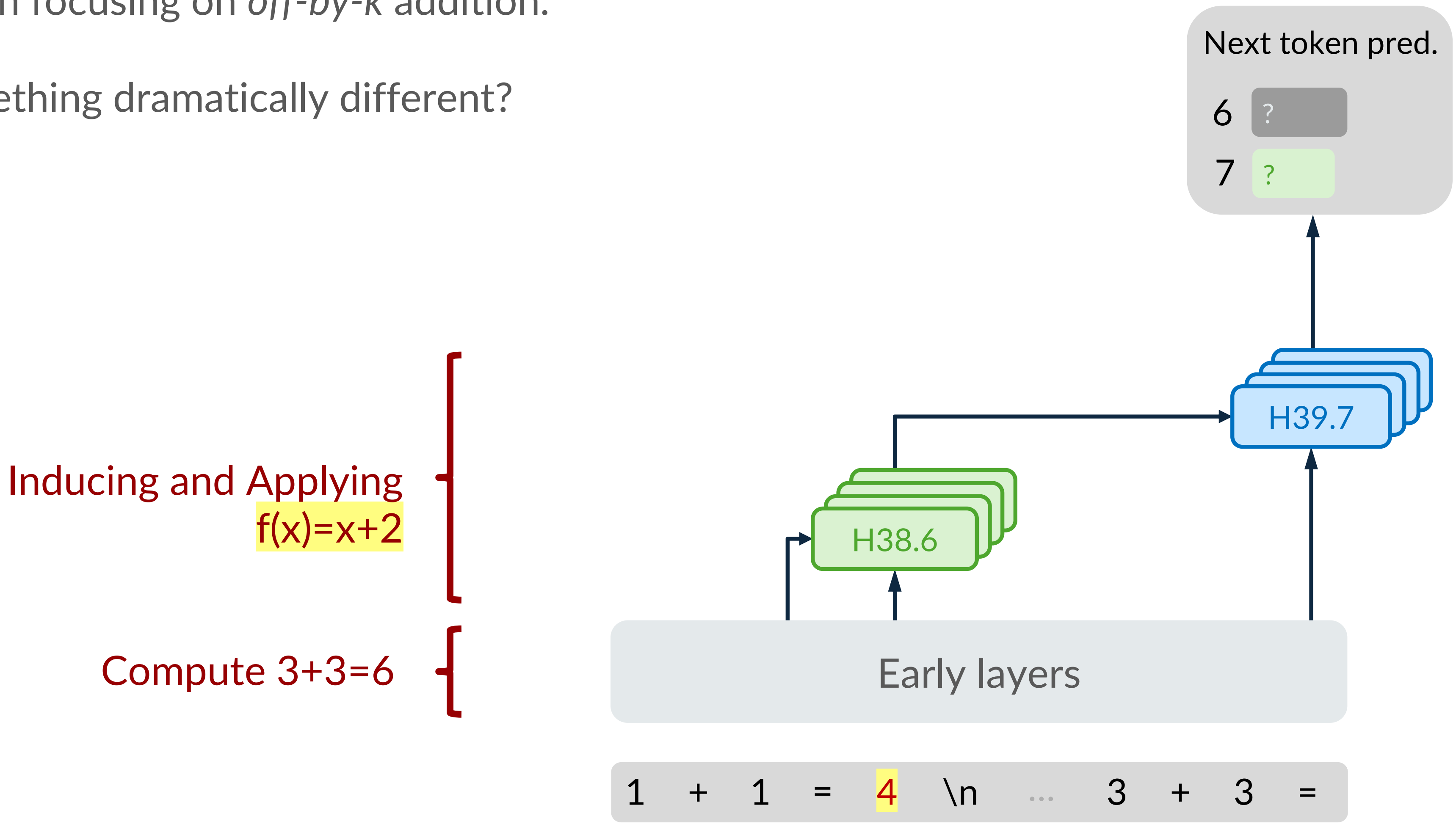
From Off-by-one to Off-by-k Addition

- This observation is consistent with different offsets.
- When FI heads are present, the model performs off-by-k addition non-trivially.
- When FI heads are ablated, the model performs standard addition instead.



From Off-by-k Addition to More

- So far, we've been focusing on *off-by-k* addition.
- What about something dramatically different?

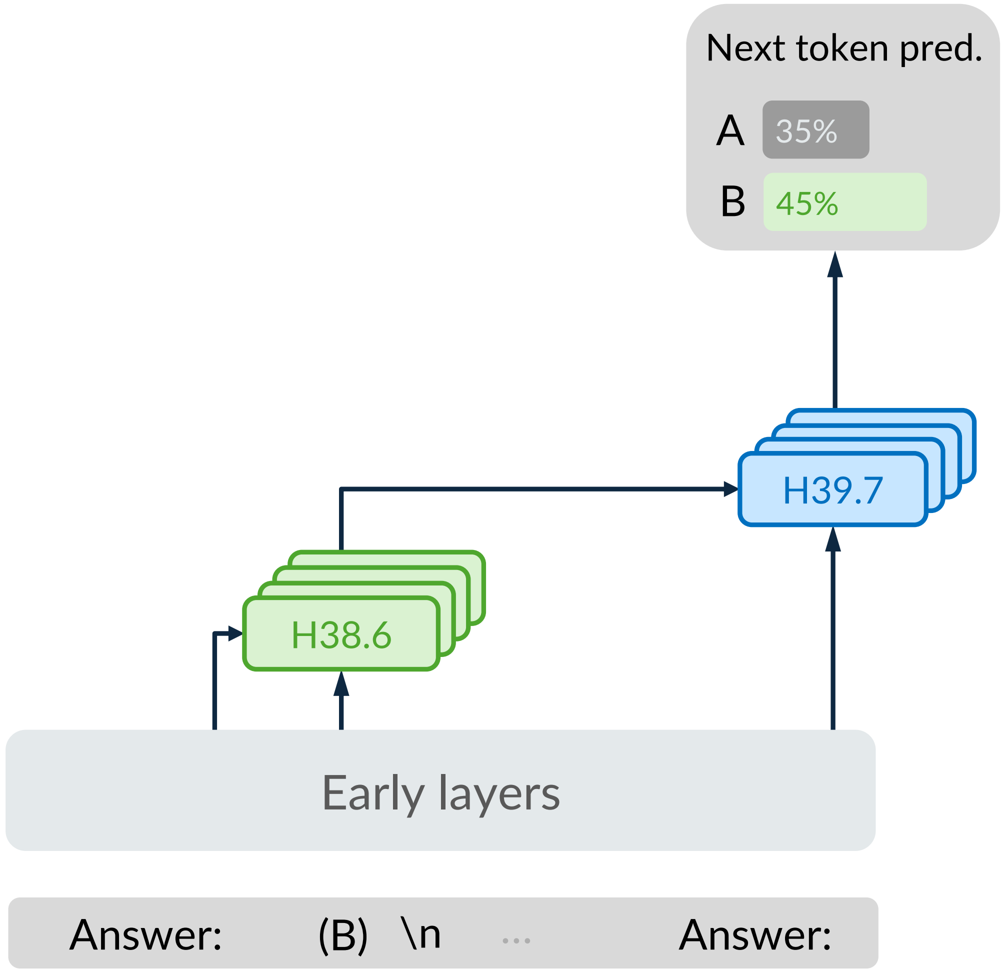


From Off-by-k Addition to More

- So far, we've been focusing on *off-by-k* addition.
- What about something dramatically different?

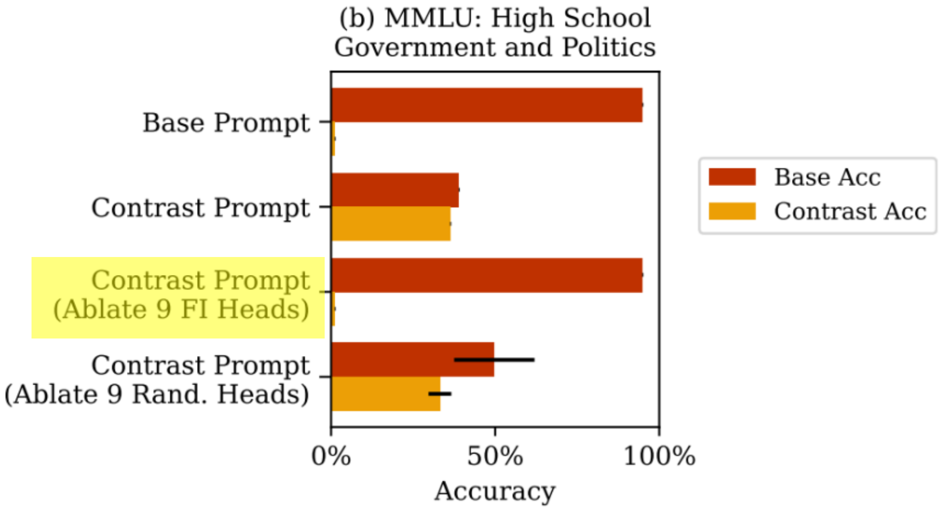
Shifting the answer by one
(A→B, B→C, ...)

Multiple-choice QA



Finding 3: Function Induction Helps Task Generalization

- The same set of FI heads are reused in Shifted MMLU.
    - When FI heads are present, the model performs Shift-by-one MMLU.
    - When FI heads are ablated, the model performs Standard MMLU.



Finding 3: Function Induction Helps Task Generalization

- We tried more tasks! The same set of FI heads are reused in Caesar Cipher and Base-k Addition.
- We took a closer look at base-8 addition.

Base-10	25+16=41\n60+16=76\n13+35=48\n52+17= **69**
Base-8	25+16=43\n60+16=76\n13+35=50\n52+17= **71**

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Case 1
Base-10 and base-8 answers are the same.

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Case 2
Unit digit `c[0]+=2`
Eights digit `c[1]+=1`

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Eights digit `c[1]+=1`

Base-10	25+16=41	60+16=76	13+35=48	52+17= **69**
Base-8	25+16=43	60+16=76	13+35=50	52+17= **71**

Case 1
Base-10 and base-8
answers are the same.

Finding 3: Function Induction Helps Task Generalization

Case 3
Unit digit `c[0]+=2`

Case 2
Unit digit `c[0]+=2`
Eights digit `c[1]+=1`

Base-10	25+16=41	60+16=76	13+35=48	52+17= **69**
Base-8	25+16=43	60+16=76	13+35=50	52+17= **71**

Case 1
Base-10 and base-8
answers are the same.

- We generate 100 test examples for each category.

- The model uses FI heads to apply +1 and +2;
- But does not always apply them under the right conditions.

Summary: Function Induction

- We interpret how models perform **off-by-one addition**.

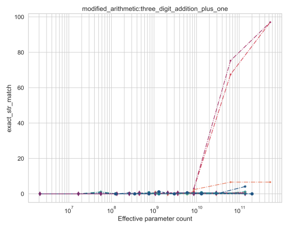
- LMs implement a complex **function induction** mechanism.
  - Leveling up from token-level copy-paste induction.
- Function induction heads work **collaboratively**.
  - Each send out a fraction of “+1”, which adds up to the whole “+1” function.
- The function induction mechanism **helps task-level generalization** broadly.
  - Components in off-by-one addition are reused in off-by-k addition, shifted MMLU, base-k addition ...

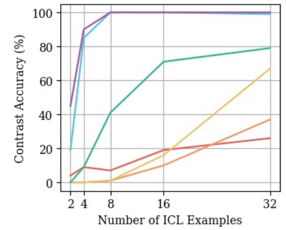
Research Question: Revisited

Llama 3

1+1=3
2+2=5

PaLM





How do LMs perform off-by-one addition?

Can models learn unseen tasks with ICL?

How do LMs handle misinformation?

Why do emergent abilities emerge?

Research Question: Revisited

Can models learn unseen tasks with ICL?

- Speculation
  - If an unseen task can be viewed as a seen task + a simple function.
  - The language model may be able to compose them together via in-context learning.

Unseen Task
Off-by-one Addition

Task: $f(g(a,b))=a+b+1$

Subtask 1: $g(a,b)=a+b$ ← seen task

Subtask 2: $f(x)=x+1$ ← function

Research Question: Revisited

How do LMs handle misinformation?

- Speculation
  - Models (investigated in this work) tend to not only follow 1+1=3, but also generalize it to 2+2=5.

Research Question: Revisited

Why do emergent abilities emerge?

- Speculation
  - For two-step tasks, early layers in the LM perform step 1, and late layers perform step 2.
  - Smaller models may not have enough layers (capacity) to develop this sequential structure.

Future directions

How does the function induction mechanism form during pre-training?

- Speculation
  - **FI heads** may evolve from induction heads (Olsson et al., 2022) and function vector heads (Todd et al., 2023).
- It will be interesting to
  - Reproduce our results using an open model (e.g., OLMo 2);
  - Examine the mechanism with intermediate checkpoints;
  - Conduct a study similar to Yin et al., 2025.

Which Attention Heads Matter for In-Context Learning?

Kayo Yin [1] Jacob Steinhardt [1]

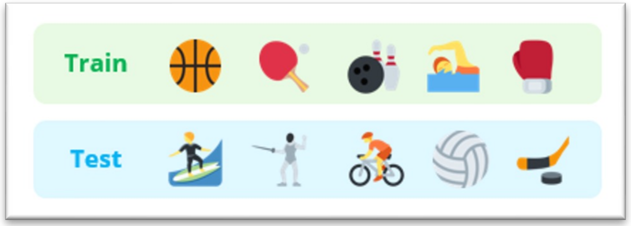
Future directions

How is function induction reused in naturally-occurring text?

- Our work is currently limited to synthetic tasks and algorithmic tasks.
- It will be interesting to
  - Disable the function induction mechanism in the model;
  - Search for sentences where it has maximal impact.

My PhD Journey

- During my PhD, I worked on **cross-task generalization abilities of large language models**.
  - **Measuring** cross-task generalization by training language models across diverse NLP tasks.
  - **Predicting** cross-task generalization through data-driven modeling and analysis.
  - **Deconstructing** cross-task generalization by dissecting model internals and uncovering underlying mechanisms.

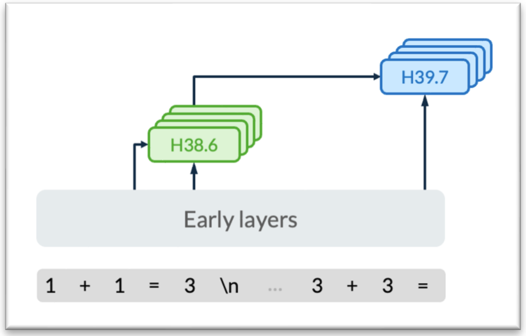


CrossFit
(EMNLP 2021)

Model Family	# param	Task	# shot	Perf.
GPT-3	3B	strategy_qa	0	0.48
BIG-G T=1	8B	elementary_math	3	0.19
PaLM	64B	code_line_desc	2	0.23
GPT-3	6B	elementary_math	1	?

How *predictable* are LLM capabilities?

BIG-bench Analysis
(EMNLP Findings 2023)



Function Induction
(This Talk; In Submission, 2025)

Thank you!